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The Circular Number of a Graph

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Abstract:

If I_c [S]=V(G), then a set S⊆V (G) is a circular set of G. The circular number of G, represented by cr(G), is the lowest cardinality of a circular set of G. A cr-set of G is any circular set with cardinality cr(G). In this study, we determine the circular number of certain standard graphs. It is demonstrated that there exists a connected graph G such that dn(G)=a, g(G)=b and cr(G)=c for each integer a,b, and c with a>2,b>2, and c>2. The corona of graphs and circular number of joins were also explored.

Keywords: distance, detour number, geodetic number, circular number, join of a graph and corona of a graph.

1. Introduction and Preliminaries

A graph $G = (V, E)$ is a finite, undirected connected graph without loops or many edges. For order and size, G is denoted by n and m, respectively. To define fundamental ideas in graph theory, we use [1,5]. Two vertices, "u and v, are taken into consideration in G if $uv \in E(G)$. $N(v)$ is the neighborhood of a vertex v in G, which is the set of vertices adjacent to v. The degree of the vertex v is $deg(v) = |N(v)|$. For any edge $e = \{u, v\}$ of a graph G where $deg(u) = 1$ and $deg(v) > 1$, we define v an end edge, u a leaf, and v a support vertex. $\Delta(G)$ indicates a graph G's maximum degree. $G[S]$ is the subgraph that a graph G with a set S of vertices induces, where $uv \in E(G)$: $u, v \in S\}$ and $V\ (G[S]) = S$. An extreme vertex of G is a vertex v if and only if $G[N(v)]$ is complete.

The length $d(u, v)$ is the shortest path length between two vertices $u, v \in V(G)$. Any $u - v$ path with length $d(u, v)$ is a $u - v$ geodesic of G. An internal vertex of a $u - v$ route P is x if it is a vertex of P and $x \neq u, v$. It is evident that the closed interval $I[u, v]$ consists of u, v and all vertices on a $u - v$ geodesic of G. A non-empty set $S \subseteq V(G)$ is closed by the set $I[S] = \bigcup_{u,v \in S} I[u, v]$. A set $S \subseteq V(G)$ is a geodetic set if and only if $I[S] = V\ (G)$. The most minimal cardinality of a geodetic set of G is its geodetic number, denoted as g(G). A minimum cardinality geodetic set of G is termed as g-set. For graph geodetic parameters, see [2,3,7]. The detour distance $D(u, v)$ is the longest path in $V(G)$ between two vertices, u and v.

Any u-v path of length $D(u, v)$ is a $u - v$ detour of G. The closed interval $I_D[u, v]$ is made up of u, v, and all of its vertices lie on a $u - v$ detour of G. If $S \subseteq V\ (G)$ is non-empty, its closure can be found in the set $I_D[S] = \bigcup_{u,v \in S} I_D[u, v]$. A set $S \subseteq V(G)$ is considered a detour set if $I_D[S] = V\ (G)$. In a detour set of G, the lowest cardinality is the detour number, denoted by $dn(G)$. A detour set of minimum cardinality of G can be identified as a dn-set. Also numerous researches covered these ideas [4,6].

$D^c(u,v)$ represents the circular distance between u and v, is defined by

$$Dc^c(uc, vc) = \begin{cases} D(uc, vc) + d(uc, vc) & \text{if } uc \neq v \\ 0 & \text{if } uc = v \end{cases}$$

where the distances between uc and vc are detour distance $D(u\text{c}, v\text{c})$ and $d(u\text{c}, v\text{c})$, respectively. The longest circle distance on G between two vertices is represented by the circular diameter D^c. An $u - v$ circular of G is any $u - v$ path of length $Dc^c(u,v)$. The longest circle distance on G between two vertices is symbolized by the circular diameter D^c. $I_c[u\text{c}, v\text{c}]$denotes the set of all vertices on a $u - v$ circular in G for any $u, v \in V$. Let $I_c[S] = \bigcup_{u,v \in S} I_c[u,v]$ for $S \subseteq V(G)$. The graph union $G_1 \cup G_2$ together with all the edges linking V_1 and V_2 is the join $G\text{c} = G_{\text{c}1} + G\text{c}_2$ of graphs G_1 and G_2 with disjoint vertices V_1 and V_2 and edge sets E_1 and E_2. Define two graphs, H and K. The graph with $V\text{c}(G\text{c} + H\text{c}) = V\text{c}(G\text{c}) \cup V\text{c}(H\text{c})$ and $E(G\text{c} + H\text{c}) = E(G\text{c}) \cup E(H\text{c}) \cup \{uv\text{c} : u\text{c} \in V\text{c}(G\text{c}), v \in V\text{c}(H\text{c})\}$ is the graph that joins G and H." The graph created by taking one copy of K and $|V(K)|$ copies of H and attaching all the vertices from the i^{th}-copy of H to the i^{th}-vertex of K by an edge, where i = 1, 2, ..., $|V\ (H\text{c})|$ is known as the corona product $K \odot H$.

2. The circular number of a graph

Definition 2.1. If $I_c[S] = V(G)$, then a set $S \subseteq V\ (G)$ is a circular set of G. The circular number of G, represented by $cr(G)$, is the lowest cardinality of a circular set of G. A cr-set of G is any circular set with cardinality $cr(G\text{c})$.

Example 2.2. For the graph G as illustrated in Figure 2.1, a cr-set of G is S= $\{v_1, v_5, v_{10}\}$, such that $cr(G) = 3$.

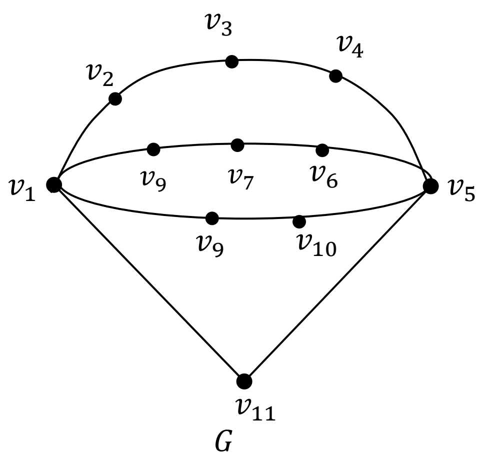

Figure 2.1

Observation 2.3. (i) $2 \leq c_r(G) \leq n$ for a connected graph of order $n \geq 2$.
(ii) In a connected graph G, every circular set of G has an end vertex that belongs to it.

(iii) No cr-set of G contains a cut vertex of G.

Theorem 2.4. For the complete graph $G\text{c} = K_{n\text{c}}$ $(n\text{c} \geq 3)$, $cr(G\text{c}) = 2$.

Proof. Assume that G has two vertices, $\{x, y\}$. $I_c[x, y] = V(G)$ in that case. Consequently, $cr(G) = 2$ since $\{x\text{c}, y\text{c}\}$ is a circular set of G.

Theorem 2.5. For the complete bipartite graph $G = K_{r,s}$, $(1 \leq r \leq s)$, $cr(G) = 2$.

Proof. Let X and Y be the two bipartite sets of G. Let $S = \{x\text{c}, y\text{c}\}$ where $x \in X$ and $y\text{c} \in Y$. Then $I_c[x, y\text{c}] = V(G)$. Hence S is a circular set of G so that $cr(G\text{c}) = 2$.

Theorem 2.6. For the star $G\text{c} = K_{\text{c}1,n\text{c}-1}$ $(n \geq 3)$, $cr(G\text{c}) = n - 1$.

Proof. This is inferred from Observation 2.3 (ii) and (iii).

Theorem 2.7. For the wheel graph $G = K_1 + C_{n-1}$ $(n\text{c} \geq 4)$, $cr(G\text{c}) = 2$.

Proof. Let $V(K_1) = x$, $V(C_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$. Let $S = \{x, v_1\}$. Then $I_c[S] = V(G)$. Therefore S is a circular set of Gc so that $cr(G) = 2$.

Theorem 2.8. For the fan graph $G\text{c} = K_{1\text{c}} + P_{n-1}$ $(n\text{c} \geq 4)$, $cr(\text{c}G) = 2$.

Proof. Let $V(K_{1\text{c}}) = x$ and $V(P_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$ let $S = \{x, v_1\}$. Then $I_c[S] = V(G)$ and so S is a circular set of Gc. Therefore $cr(G\text{c}) = 2$.

Theorem 2.9. Consider the connected graph G, which has a detour diameter D, a circular diameter D^c, and a diameter d. Let P:u-v represent the "circular diameter of G. Let P_1 and P_2 be a diametral path and detour diametral path of G. Such that

$$cr(G) \leq \begin{cases} n\text{c} - d + 1 & \\ n - D + 1 & if\ V(P_{1\text{c}}) = V(P_{2\text{c}}) \\ n - D^c + 1 & V(\text{c}P_1) \cap V(\text{c}P_2) = \{u\text{c}, v\text{c}\} \\ n - D^c + 1 + d + D & V(\text{c}P_1) \cap V\text{c}(P_2) \neq \{\text{c}u, v\} \end{cases}$$

Proof. Let P: $u_0, u_1, u_2, \dots\dots u_{D^c} = v$ be a circular diametral path of G, $P_1: x\text{c} = x_{0\text{c}}, \text{c}x_1, \text{c}x_2, \dots, x_r = y$ be a diametral path of G and $P_2: w = w_0, w_1, w_2, \dots, w_s = z$ be a asedetour set of G. Then r+s=d+D.

Case 1: Let $V(P_1) = V(P_2)$. Then $S = V(G) - \{\text{c}x_1, \text{c}x_2, \dots, x_{r-1}\}$ is a circular set of G, so that $cr(G) \leq n - (r - 1) = n - r + 1 = n - d + 1$. By the similar way we can prove that $cr(G) \leq n - D + 1$.

Case 2: $V\text{c}(P_1) \cap V\text{c}(P_2) = \{u\text{c}, v\text{c}\}$. Then $S = V\text{c}(G\text{c}) - \{u_1, u_2, \dots\dots u_{D^c-1}\}$ is a circular set of G, so that $cr(G) \leq n - D^c + 1$

Case 3: $V(P_1) \cap V(P_2) \neq \emptyset \neq \{u, v\}$ .Without loss of generality let us assume

that $x_1', x_2', \dots, x_r'$ in P_1 and $w_1', w_2', \dots, w_s'$ *which* is not belongs to $V(P_1) \cap V(P_2)$. Then $S = V(G) - \{\text{c}u_1, \text{c}u_2, \dots\dots u_{D^c-1}\} \cup \{\text{c}x_1', \text{c}x_2', \dots, \text{c}x_r'\} \cup \{\text{c}w_1', \text{c}w_2', \dots, w_s'\}$ is a circular set of cG, so that $cr(G) \leq n - D^c + 1 + r + s \leq n - D^c + 1 + d + D$.

Theorem 2.10. Let G be a non-complete connected graph of order $n \geq 3$, and its vertex connectivity is denoted as $\kappa(G)$ in $cr(G) \leq n - \kappa(G)$.

Proof. Since Gc is non-complete, it follows that 1≤ $\kappa(G) \leq n - 2$. Let $Z = \{z_1, z_2, \ldots, z_\kappa\}$ be the minimal cutset of the vertices of G. Clearly $S = V(G) - Z$ where cG_1, c$G_2, \ldots, G$c$_r$ (rc ≥ 2) be the components of Gc– Uc. As a result, every vertex u_i (1≤ $i \leq \kappa$) has at least one neighbouring vertex in G_j ($1 \leq j$c $\leq r$c). Thus $I_c[S] = V(G)$ implies that S is a circular set of G. As a result, $cr(G) \leq n - \kappa(G)$.

Next, we demonstrate that for every integer $n \geq 4$, there exists a single connected network of rank n with circular number $n - 1$.

Theorem 2.11. A connected graph G of order p, detour number k and detour diameter D exist for every triple D, k, p of integers with $2 \leq k \leq p - D^C D + 1$ and $D \geq 2$.
Proof. Let G represent the graph that was created from the cycle C_{D^C-1}: $u_1, u_2, \ldots\ldots u_{D^C-1}, u_1$ of order $D^C - 1$ by (1) integrating $k - 1$ new vertices to $v_1, v_2, \ldots\ldots, u_{k-1}$ and combining each vertex v_i ($1 \leq i \leq k-1$) to u_1 and (2) merging $n - D^C - k + 2$ new vertices to $w_1, w_2, \ldots\ldots, w_{n-D-k+2}$ and incorporating every vertex w_i ($1 \leq i \leq n - D^C - k + 2$) to both u_1 and u_3." The graph G, which is displayed in Figures 2.2 and 2.3, has order n and a circular diameter of D^C.

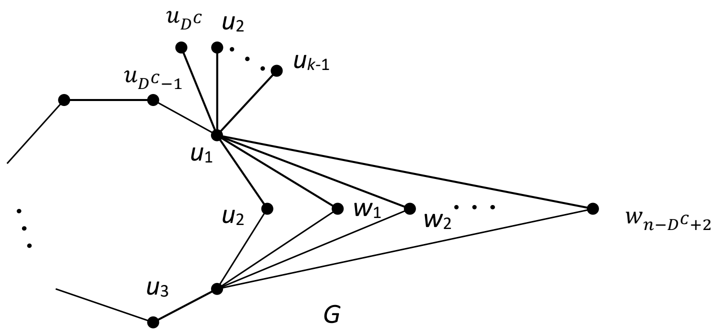

Figure 2.2

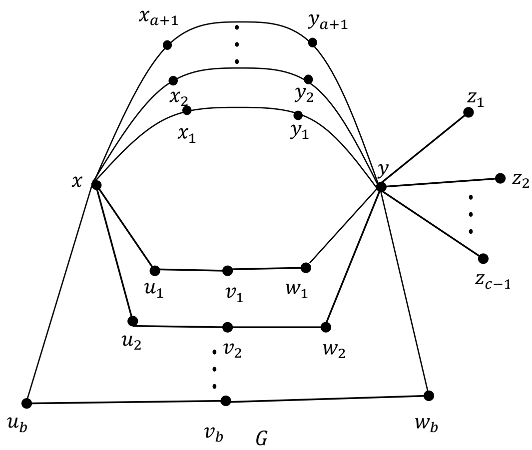

Figure 2.3

3. Circular number of Join of two graphs

Theorem 3.1. Let $G = K_{n_1} + H$ ($n \geq 2$) where H is non-complete connected graph of order $n_2 \geq 2$. Then $cr(G$c$) \geq cr(H$c$)$.

Proof. Let G be a connected path with $V(K_{n_1}) = \{v_1, v_2, \ldots, v_{n_1}\}$ ($n_1 \geq 2$) and Hc be a non-complete connected graphs of order $n_2 \geq 2$. Let S be a cr-set of cH. If $D^c(u, v) \leq 3$ for every $u, x \in S$, then cS is a circular set of Gc so that $cr(G$c$) = |S| = cr(H$c$)$.

If $D_H^c(u,v) \geq 4$ for some $u, v \in S$, then $d_G(x,y) \leq 2$ for every $x\text{c}, y\text{c} \in G\text{c}$ and $D_{G\text{c}}(x\text{c}, y) \geq 2$ for every $x\text{c}, y\text{c} \in G\text{c}$, $S\text{c}$ is not a circular set of $G\text{c}$. Let S_1 be a cr-set of $G\text{c}$. Since for all $u, v \in S_1$, $D_H^c(u,v) \geq 4$, $|S_1| > |S|$. Therefore $cr(G) \geq cr(H)$.

Theorem 3.2 $cr(K_{n_1} + P_{n_2}) = 2$, where $n_1, n_2 \geq 2$.

Proof. Let $V\text{c}(K_{n_1})$ be $u_1\text{c}, u_{2\text{c}}, \dots, u_{n_1}$ and $V(P_{n_2})$ be $v_1, v_2, \dots, v_{n_2}$. Let $S\text{c} = \{u_{1\text{c}}, v_{n_2}\}$. Then S is a circular set of G so that $cr(K_{n_1} + P_{n_2}) = 2$.

Theorem 3.3 $cr(K_{n_1} + K_{n_2}) = 2$, where $n_1, n_2 \geq 2$.

Proof. Since $K_{n_1} + K_{n_2}$ is the complete graph $K_{n_1+n_2}$, then Theorem 2.4 yields the desired outcome.

Theorem 3.4 $cr(C_{n_1} + C_{n_2}) = 2$, where $n_1, n_2 \geq 3$.

Proof. Let $V(C_{n_1})$ be $u_{1\text{c}}, u_{2\text{c}}, \dots, u_{n_1}$ and $V(C_{n_2})$ be $v_1, v_2, \dots, v_{n_2}$. Let $S\text{c} = \{u_{1\text{c}}, v_{n_2}\}$. Then S is a circular set of G so that $cr(C_{n_1} + C_{n_2}) = 2$.

Theorem 3.5 Let P_{n_1} and P_{n_2} be two non-trivial paths of order $n_1 \geq 2$ and $n_2 \geq 2$ respectively. Then $cr(P_{n_1} + P_{n_2}) = 2$.

Proof. Let $V\text{c}(P_{n_1})$ be $u_{1\text{c}}, u_{2\text{c}}, \dots, u_{n_1}$ and $V(P_{n_2})$ be $v_1, v_2, \dots, v_{n_2}$. Let $S\text{c} = \{u_{1\text{c}}, v_{n_2}\}$. Then $S\text{c}$ is a circular set of $G\text{c}$ so that $cr(P_{n_1} + P_{n_2}) = 2$.

Theorem 3.6 Let $cr(\overline{K}_{n_1} + H) = 2$, where $n_1 \geq 1$ and H is a connected graph of order $n_2 \geq 2\text{c}$.

Proof. Let $V\text{c}(\overline{K}_{n_1})$ be $u_1, u_2, \dots, u_{n_1}$ and H be $v_1, v_2, \dots, v_{n_2}$. Let $S = \{\text{c}u_1, v_{n_2}\}$. Then $S\text{c}$ is a circular set of $G\text{c}$ so that $cr(\overline{K}_{n_1} + \text{c}H) = 2$.

4. Circular number of Corona of two graphs c

Theorem 4.1. $cr(C_{n_1} \circ C_{n_2}) = 2n_1 - 2$, $n_1 \geq 3$ and $n_2 \geq 3$.

Proof. Let $V(P_{n_1}) = \{u_1, u_2, \dots, u_{n_1}\}$ and $V(P_{n_2}) = \{v_1, v_2, \dots, v_{n_2}\}$. The set of vertices of i^{th} of P_{n_2} is referred as $V(P_{n_2}^i) = \{v_1^i, v_2^i, \dots, v_{n_2}^i\}$ $(1 \leq i \leq n_1)$. Since every circular set of G contains at least one vertex from $P_{n_2}^1$ and $P_{n_2}^{n_1}$ are at least two vertices from $P_{n_2}^i$ $(2 \leq i \leq n_1 - 1)$, $cr(P_{n_1} \circ P_{n_2}) \geq 2 + 2$ $(\quad n_1 - 1) = 2n_1 - 2$ $\quad cr(P_{n_1} \circ P_{n_2}) = 2$ $(\quad n_1 - 1)$. Therefore $S = \{v_1^1, v_{n_2}^{n_1}, v_1^2, v_{n_2}^2, v_1^3, v_{n_2}^3, \dots, v_1^{n_1-1}, v_{n_2}^{n_1-1}\}$, such that S is a circular set of G. Hence $cr(P_{n_1} \circ P_{n_2}) = 2n_1 - 2$.

Theorem 4.2. $cr(P_{n_1} \circ P_{n_2}) = 2(n_1 - 1)$ where $n_1 \geq 2$ and $n_2 \geq 2$.

Proof. Let $V(P_{n_1}) = \{u_1, u_2, \dots, u_{n_1}\}$ and $V(P_{n_2}) = \{v_1, v_2, \dots, v_{n_2}\}$. Let $V(P_{n_2}^i) = \{v_1^i, v_2^i, \dots, v_{n_2}^i\}$ $(1 \leq i \leq n_1)$ be the set of vertices of i^{th} of P_{n_2}. Since $P_{n_2}^1$ and $P_{n_2}^{n_1}$ are at least two vertices from $P_{n_2}^i$ $(1 \leq i \leq n_1)$, every circular set contains at least one vertex from each of these sets; so, $cr(P_{n_1} \circ P_{n_2}) \geq 2 + 2\,(n_1 - 1) = 2n_1 - 2$.

Let $S = \{v_1^1, v_{n_2}^{n_1}, v_1^2, v_{n_2}^2, v_1^3, v_{n_2}^3, \ldots, v_1^{n_1-1}, v_{n_2}^{n_1-1}\}$. Therefore S is a circular set of G such that $cr(P_{n_1} \circ P_{n_2}) = 2n_1 - 2$. c

Theorem 4.3. Let $cr(H \circ \overline{K}_{n_2}) = n_1 n_2, n_2 \geq 1$ and H be any non-trivial connected graph for $n_1 \geq 2$.

Proof. Let $V(H\text{c}) = \{u\text{c}_1, \text{c}u_2, \ldots, \text{c}u_{n_1}\}$ and $V(\overline{K}_{n_2}) = \{\text{c}v_1, v\text{c}_2, \ldots, v_{n_2}\}$. Let $V(\overline{K}_{n_2}^i) = \{v_1^i, v_2^i, \ldots, v_{n_2}^i\}$ $(1 \leq i \leq n_1)$ be the set of vertices of i[th] of $\overline{K}_{n_2}$. Thus the set of end vertices of G is represented as $S = \bigcup_{i=1}^{n_1} V(\overline{K}_{n_2}^i)$. According to Theorem 1.1, S is a subset for every circular set of G and so $cr(H \circ \overline{K}_{n_2}) \geq n_1 n_2$. Hence $I_{D^c}[S] = V(G)$, S is a circular set of $H\text{c} \circ \overline{K}_{n_2}$. Therefore $cr(H\text{c} \circ \overline{K}_{n_2}) = n_1 n_2$. c

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