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# Induced H-Packing k-Partition Problem in Certain Networks

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***Abstract:** A collection $\mathcal{K}$ = {$H_1$,$H_2$,..., $H_r$ } of induced sub graphs of a graph G is said to be sg-independent if (i) $V(H_i) \cap V(H_j) = \Phi$, $i \neq j$, $1 \leq i$, $j \leq r$ and (ii) no edge of G has its one end in $H_i$ and the other end in $H_j$ , $i \neq j$, $1 \leq i, j \leq r$. If $H_i \simeq H$, $\forall i$, $1 \leq i \leq r$, then $\mathcal{K}$ is referred to as a H-independent set of G. Let H be a perfect or almost perfect H-packing of a graph G. Finding a partition $\{H_1, H_2, \ldots H_k\}$ of $\mathcal{H}$ such that $\mathcal{H}_i$ is H-independent set, $\forall i$, $1 \leq i \leq k$, with minimum k is called the induced H-packing k-partition problem of G. The induced H-packing k-partition number denoted by ipp(G,H) is defined as ipp(G,H) = min $ipp_{\mathcal{H}}$ (G,H) where the minimum is taken over all H-packing of G. In this paper we obtain the induced H-packing k-partition number for Enhanced hypercube, Augmented Cubes and Crossed Cube networks where H is isomorphic to $P_3$ and $C_4$.*

***Keywords:** Augmented Cubes, Crossed Cube Networks, Enhanced hypercube, Induced H-packing k-partition.*

## I. INTRODUCTION

For any graph *G*, let *V*(*G*) denote the set of vertices in *G* and *E*(*G*) denote the set of edges in *G*, |*V*(*G*)| and |*E*(*G*)| denote the respective cardinalities of these sets. An *H-packing* of a graph *G* = (*V*, *E*) is a set of vertex disjoint sub graphs of *G*, each of which is isomorphic to a fixed graph *H*. A *perfect H-packing* in a graph *G* is a set of *H*-subgraphs of *G* such that every vertex in *G* is incident with one *H*-subgraph in this set. An almost perfect *H*-packing in a graph *G* is a set of *H*-subgraphs of *G* such that at most $|V(H)| - 1$ number of vertices are not incident on any *H* - subgraph in *G* [13], [14]. We define this concept as follows: A collection $\mathcal{K} = \{H_1, H_2, \ldots, H_r\}$ of induced sub graphs of a graph *G* is said to be *sg-independent* if (i) $V(H_i) \cap V(H_j) = \Phi$, $i \neq j$, $1 \leq i$, $j \leq r$ and (ii) no edge of *G* has its one end in $H_i$ and the other end in $H_j$, $i \neq j$, $1 \leq i$, $j \leq r$. If $H_i \simeq H$, $\forall i$, $1 \leq i \leq r$, then $\mathcal{K}$ is referred to as a *H-independent set* of *G*. Let $\mathcal{H}$ be a perfect or almost perfect *H*-packing of a graph *G*. Finding a partition $\{H_1, H_2, \ldots \mathcal{H}_k\}$ of *H* such that $H_i$ is *H*-independent set, $\forall i$, $1 \leq i \leq k$, with minimum *k* is called the induced *H*-packing *k*-partition problem of *G*. The minimum induced *H*-packing *k*-partition number is denoted by $ipp_H(G,H)$. The induced *H*- packing *k*-partition number denoted by *ipp*(*G*, *H*) is defined as *ipp*(*G*, *H*) = min $ipp_{\mathcal{H}}(G, H)$ where the minimum is taken over all *H*-packing of *G*. Packing is an extension of matching. An Induced matching and induced matching partitions of certain interconnection networks was studied [2], [11]. The induced *H*-packing *k*-partition problem was studied for certain interconnection networks such as hypercubes, Sierpiński graphs [12]. An approximation algorithm for maximum $P_3$-packing in subcubic graphs was studied by Kosowski et al [10]. Xavier et al [12] proved that the induced $P_3$-packing *k*-partition problem is *NP*- complete, also induced $C_4$-packing *k*-partition problem is *NP*-complete. In this paper we obtain the induced *H*-packing *k*-partition number for Enhanced hypercube, Augmented Cubes and Crossed Cube networks where *H* is isomorphic to $P_3$ and $C_4$.

## II. ENHANCED HYPERCUBE NETWORKS

A Hypercube with extra connections called skips is referred to as an enhanced hypercube. As a variant of the $Q_n$, enhanced hypercubes $Q_{n,k}$ ($n \geq 2$, ($1 \leq k \leq n-1$) are proposed to improve the efficiency of the hypercube architecture and have found substantial applications. Inherited from $Q_n$, $Q_{n,k}$ is also a regular graph [15], [20]. But the enhanced hypercubes are much more attractive than normal hypercubes due to its potential nice topological properties. The enhanced hypercube $Q_{n,k}$ ($1 \leq k \leq n-1$), is a graph with vertex set $V(Q_{n,k}) = V(Q_n)$ and edge set $E(Q_{n,k}) = E(Q_n) \cup$ $(x_0x_1x_2, \ldots, x_{k-2}, x_{k-1}, x_k \ldots x_{n-1}, x_0x_1x_2 \ldots x_{k-2}, \bar{x}_{k-1}, \bar{x}_k \ldots \bar{x}_{n-1})$. The edges of $Q_n$ in $Q_{n,k}$ are hypercube edges and the remaining edges of $Q_{n,k}$ are called complementary edges [4], [7], [8], [9], [16], [17], [18], [20], [21].When $k=0$, $Q_{n,0}$ reduces to the *n*-dimensional hypercube. The enhanced hypercubes $Q_{n,k}$ ($1 \leq k \leq n-1$) proposed by Tzeng and Wei [15] are $(n+1)$ regular. They have $2^n$ vertices and $(n+1)2^{n-1}$ edges.

**Theorem 1.1.** Let *G* be the Enhanced hypercube network $Q_{n,2}$ $n \geq 2$, Then *G* has an almost perfect $P_3$-packing.

*Proof.* We prove the result by induction on the dimension *n* of the Enhanced hypercube network $Q_{n,2}$. We begin with $n = 2$, $\mathcal{P}^2 = \{(00, 10, 11)\}$ is an almost perfect $P_3$-packing leaving out one vertex unsaturated. In $Q_{3,2}$, $\mathcal{P}^3 = 0\mathcal{P}^2 \cup 1\mathcal{P}^2$ is an almost perfect $P_3$-packing leaving out two unsaturated vertices inducing an edge, where $i\mathcal{P}^2$ denotes the

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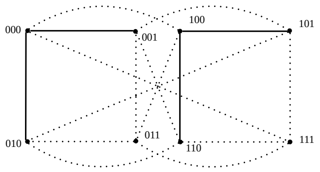
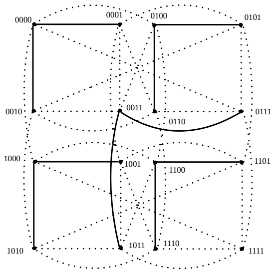
set of paths in $\mathcal{P}^2$ prefixed by $i$, $i = 0, 1$. See Fig. 1(a). Assume the result to be true for $Q_{n,2}$. Consider $Q_{n+1,2}$. Suppose $n + 1,2$ is even. By induction hypothesis $\mathcal{P}^{r+1}$= $0\mathcal{P}^r \cup 1\mathcal{P}^r$ is an almost perfect $P_3$-packing leaving out two unsaturated vertices in each copy of $Q_{n,2}$ in $Q_{n+1,2}$. By construction the four left out vertices induce a cycle $C$. Let $P$ be a sub path of length 2 in $C$. Then $\mathcal{P}^{r+1}$=$0\mathcal{P}^r \cup 1\mathcal{P}^r \cup P$ is an almost perfect $P_3$-packing leaving out one vertex unsaturated. Suppose $n +1,2$ is odd. Since $n$ is even, each copy of $Q_{n,2}$ in $Q_{n+1,2}$ contains an almost perfect $P_3$-packing leaving out one vertex unsaturated. The union is an almost perfect $P_3$-packing leaving out two unsaturated vertices in $Q_{n+1,2}$.




(a)




(b)

**Fig. 1. (a) An induced $P_3$-packing 2-partition number of $Q_{3,2}$ (b) An induced $P_3$-packing 3 - partition number of $Q_{4,2}$.**

**Lemma 1.2**. $ipp(Q_{3,2}, P_3) = 2$.

Proof. Let $P$: $p_1p_2p_3$ be a path of length 2 in $Q_{3,2}$. Then $|\cup_{i=1}^{3} N(p_i)| = 5$. Hence consider another path $Q$ of length 2 such that $V(P) \cap V(Q) = \emptyset$ contains at least two vertices from $\cup_{i=1}^{3} N(pi)$. This implies that $ipp(Q_{3,2}, P_3) \geq 2$. Now let $P$ ={(010, 000, 001)} and $Q$={(110, 100, 101)}. $P \cup Q$ is an optimal induced $P_3$-packing 2-partition leaving out two vertices unsaturated in $Q_{3,2}$.

**Lemma 1.3.** $ipp(Q_{4,2}, P_3) \geq 3$.

*Proof.* $Q_{4,2}$ is packed with 5 vertex disjoint paths of length 2, leaving out one vertex unsaturated. Suppose $ipp(Q_{4,2}, P_3) = 2$. Let $[V_1]$ and $[V_2]$ be the induced $P_3$-packing 2-partition sets. There are two possibilities. (i) $\|[V_1]\|=4$, $\|[V_2]\|=1$ and (ii) $\|[V_1]\| = 3$, $\|[V_2]\| =2$. We claim that $\|[V_1]\| \geq 3$ is not possible. Suppose $\|[V_1]\|= 3$. Let $P : uvw$ be in $[V_1]$. Then $|N(u) \cup N(v) \cup N(w)|$=8. Now $V(P)$ and its neighboring vertices constitute 11 vertices leaving 5 vertices unsaturated. If $Q$ and $R$ are the other two paths of length 2 in $[V_1]$, then $N(V(P))=N(V(Q))=N(V(R))$, a contradiction. If $\|[V_1]\|=3$ is not possible, $\|[V_1]\|>3$ is also not possible. This implies that $ipp(Q_{4,2}, P_3) \geq 3$. Now let $P$ = {(0010, 0000, 0001), (1110, 1100, 1101)}, $Q$ = {(0110, 0100, 0101), (1010, 1000, 1001)} and $R$ = {(1011, 0011, 0111)}. $P \cup Q \cup R$ is an optimal induced $P_3$-packing 3-partition leaving out one vertex unsaturated in $Q_{4,2}$.

**Lemma 1.4.** $ipp(Q_{6,2}, P_3) \geq 3$.

*Proof.* $Q_{6,2}$ contains four copies of $Q_{4,2}$, say $(Q_{4,2})_i$, $1 \leq i \leq 4$. By Lemma 1.3, $ipp(Q_{4,2}, P_3) \geq 3$. Let $[V_1^i]$, $[V_2^i]$, $[V_3^i]$ be the induced $P_3$-packing 3-partition sets of $(Q_{4,2})_i$, $1 \leq i \leq 4$. One vertex $u_i$, $1 \leq i \leq 5$ in each $(Q_{4,2})_i$, $1 \leq i \leq 4$ is not included in any of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $1 \leq i \leq 4$. For optimal induced $P_3$-packing 3-partition, it is necessary that the sub graph induced by $u_1$, $u_2$, $u_3$ and $u_4$ contains a path of length 2 in $Q_{6,2}$. Consider $u_1$ in $(Q_{4,2})_1$ with $deg(Q_{4,2})_1(u_1)$= 5. If $u_1$ is adjacent to vertices in $[V_1^1]$, $[V_2^1]$ and $[V_3^1]$ then the $P_3$-path containing $u_1$ cannot be included in any of $[V_1^1]$, $[V_2^1]$ or $[V_3^1]$ a contradiction. Suppose two vertices adjacent to $u_1$ are in $[V_1^1]$, one vertex adjacent to $u_1$ is in $[V_2^1]$ and two vertices adjacent to $u_1$ are in $[V_3^1]$ then a 3-cycle is induced by these vertices, a contradiction. See Fig. 2(a). For the same reason, $u_1$ cannot be adjacent to 5 vertices in $[V_i^1]$ With $\|[V_i^1]\| = 1$, $1 \leq i \leq 3$. Hence $u_1$ is adjacent to 5 vertices in any two of $[V_i^1]$ with $\|[V_i^1]\| = 2$, $1 \leq i \leq 3$. See Fig. 2(b). This argument is also true for $u_i$ in $(Q_{4,2})_i$, $2 \leq i \leq 4$.

Claim that the binding edges in $((Q_{4,2})_1 \cup (Q_{4,2})_2) \backslash (Q_{4,2})_1$ incident at vertices of $[V_i^1]$, $1 \leq i \leq 3$, have their other ends in exactly one $[V_j^2]$, $1 \leq j \leq 3$. Suppose if not, Let all the end vertices of binding edges incident at vertices of $[V_1^1]$, be adjacent to vertices in $[V_2^2]$ and $[V_3^3]$ also end vertices of binding edges incident at vertices of $[V_2^1]$ be adjacent to vertices in $[V_3^2]$ and $[V_1^2]$ end vertices of binding edges incident at vertices of $[V_3^1]$ be adjacent to vertices in $[V_1^2]$ and $[V_2^2]$ then no vertex in $[V_3^1]$ is adjacent to any vertex in $[V_1^2]$ and $[V_2^2]$ a contradiction. See Figure 2(c). This argument is also true for $[V_i^1]$, $i$ = 2, 3. Vertex set $V(Q_{4,2})$ can be partitioned into $[V_1]$, $[V_2]$ and $[V_3]$ such that, each of $[V_1]$, $[V_2]$ contains at most 6 vertices of $V(Q_{4,2})$ and $[V_3]$ contains at most 3 vertices of $V(Q_{4,2})$. We have $\|[V_1]\|$=2, $\|[V_2]\|$=2 and $\|[V_3]\|$=1. Let $[V_1]$ ={$P$,$Q$}, where $P$: $p_1p_2p_3$ and $Q$: $q_1q_2q_3$ are in $(Q_{4,2})_1$.Then $|\cup_{i=1}^{3} N(p_i) \cap (Q_{4,2})_2|$=3 and $|\cup_{i=1}^{3} N(q_i) \cap (Q_{4,2})_2|$=3. Hence $\cup_{i=1}^{3} N(p_i) \cap (Q_{4,2})_2$ and $\cup_{i=1}^{3} N(q_i) \cap (Q_{4,2})_2$ are not in $[V_1]$. This implies $\cup_{i=1}^{3} N(p_i) \cap (Q_{4,2})_2$ and $\cup_{i=1}^{3} N(q_i) \cap (Q_{4,2})_2$ are in $[V_2]$ and $[V_3]$ . Now let $[V_2]$= {$R$, $S$},where $R$: $r_1r_2r_3$ and $S$: $s_1s_2s_3$ are in $(Q_{4,2})_1$. Then $|\cup_{i=1}^{3} N(r_i) \cap (Q_{4,2})_2|$=3 and $|\cup_{i=1}^{3} N(s_i) \cap (Q_{4,2})_2|$=3. Hence $\cup_{i=1}^{3} N(r_i) \cap (Q_{4,2})_2$ and $\cup_{i=1}^{3} N(s_i) \cap (Q_{4,2})_2$ are not in $[V_2]$. This implies $\cup_{i=1}^{3} N(r_i) \cap (Q_{4,2})_2$ and $\cup_{i=1}^{3} N(s_i) \cap (Q_{4,2})_2$ are in $[V_3]$ and $[V_1]$ . Let $[V_3]$ = {$T$}, where $T$ : $t_1t_2t_3$ is in $(Q_{4,2})_1$. Then $|\cup_{i=1}^{3} N(t_i) \cap (Q_{4,2})_2|$=3. Hence $\cup_{i=1}^{3} N(t_i) \cap (Q_{4,2})_2$ is not in $[V_3]$. This implies $\cup_{i=1}^{3} N(t_i) \cap (Q_{4,2})_2$ is in $[V_1]$ and $[V_2]$ . Similarly $(Q_{4,2})_3$ is partitioned as in $(Q_{4,2})_2$ and $(Q_{4,2})_4$ is partitioned as in $(Q_{4,2})_1$. Let $u_1$ be the unsaturated vertex in $(Q_{4,2})_1$. Then $|N(u_1)|$ = 5. The edges incident at vertices of $N(u_1)$ are adjacent to vertices in any one of $[V_i^1]$, with $\|[V_i^1]\|$=2, $1 \leq i \leq 3$. Without loss of generality let $u_1$ be adjacent to a vertex in $[V_1^1]$, Similarly let $u_2$ be the unsaturated vertex in $(Q_{4,2})_2$. Since $|N(u_2)|$ = 5, the edges incident at vertices of $N(u_2)$ are adjacent to vertices in any one of $[V_i^2]$, with $\|[V_i^2]\|$=2, $1 \leq i \leq 3$. This implies $u_2$ is adjacent to a vertex in $[V_1^2]$. For the same reason $u_3$ is adjacent to a vertex in $[V_1^3]$, and $u_4$ is adjacent to a vertex in $[V_1^4]$. Hence the edges incident at vertex $u_i$, $1 \leq i \leq 4$ are adjacent to vertices in at most one of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $1 \leq i \leq 4$ in each $(Q_{4,2})_i$ $1 \leq i \leq 4$. This implies $u_i$, $1 \leq i \leq 4$ is adjacent to at most two of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $1 \leq i \leq 4$ in $(Q_{6,2})$. Since $(Q_{4,2})_1 \simeq (Q_{4,2})_4$ and $(Q_{4,2})_2 \simeq (Q_{4,2})_3$, the unsaturated vertices from

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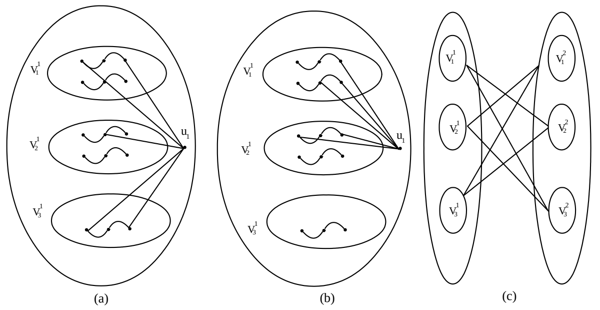

**Fig. 2. (a) and (b) Possibilities of adjacent vertices of $u_1$ (c) Possibilities of binding edges**

each $(Q_{4,2})_1$, $(Q_{4,2})_2$, $(Q_{4,2})_3$ and $(Q_{4,2})_4$ induce a vertex disjoint path of length $2$, leaving out one vertex unsaturated. This implies that the three vertices $u_2$, $u_3$ and $u_4$ are adjacent to at most two vertex sets.

Therefore $ipp(Q_{6,2}, P_3) \geq 3$.

**Lemma 1.5.** The induced $P_3$-packing $k$-partition number of $Q_{n,k}$ satisfies $ipp(Q_{n,k}, P_3) \geq \left\lfloor \frac{n}{2} \right\rfloor$, $n \geq 6$, and $k = 2$.

Proof. We prove the result by induction on the dimension $n$ of the Enhanced hypercube network $Q_{n,k}$. We prove that an unsaturated vertex $u_i$, $1 \leq i \leq 4$ in $(Q_{n-2,2})_i$, $1 \leq i \leq 4$ is adjacent to $\left\lfloor \frac{n-4}{2} \right\rfloor$ vertices in $\left\lfloor \frac{n-4}{2} \right\rfloor$ partition sets of $(Q_{n-2,2})_i$, $1 \leq i \leq 4$. We begin with $n = 8$. $Q_{8,2}$ contains four copies of $Q_{6,2}$ say $(Q_{6,2})_i$, $1 \leq i \leq 4$. By Lemma 1.4, $ipp(Q_{6,2}, P_3) \geq 3$, leaving out one vertex unsaturated. Let $[V_1^i]$, $[V_2^i]$, $[V_3^i]$ be the induced $P_3$-packing $3$-partition sets of, $Q_{6,2}$, $1 \leq i \leq 4$. One vertex $u_i$, $1 \leq i \leq 4$ in each $Q_{6,2}$, $1 \leq i \leq 4$ is not included in any of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $1 \leq i \leq 4$. For optimal induced $P_3$-packing $\left\lfloor \frac{n}{2} \right\rfloor$ partition, it is necessary that the sub graph induced by $u_1$, $u_2$, $u_3$ and $u_4$ contains a path of length $2$ in $Q_{8,2}$. Consider $u_1$ in $(Q_{6,2})_1$, $degQ_{(6,2)1}(u_1) = 7$. If $u_1$ is adjacent to vertices in $[V_1^1]$, $[V_2^1]$, $[V_3^1]$, then the $3$-path containing $u_1$ cannot be included in any of $[V_1^1]$, $[V_2^1]$, $[V_3^1]$ a contradiction. Suppose $u_1$ is adjacent to vertices in any one of $[V_i^1]$ $1 \leq i \leq 3$, then $ipp(Q_{6,2}, P_3) \geq 3$ a contradiction. Hence $u_1$ is adjacent to $7$ vertices in at most two of $[V_i^1]$, $1 \leq i \leq 3$. This argument is also true for $u_i$ in $(Q_{6,2})_i$, $2 \leq i \leq 4$. This implies that the three vertices $u_1$, $u_2$ and $u_3$ are adjacent to at most three vertex sets. This implies $ipp(Q_{8,2}, P_3) \geq \left\lfloor \frac{n}{2} \right\rfloor$. Assume the result is true for Enhanced hypercube with dimension less than or equal to $n-1$. Consider $Q_{n,k}$. When $n$ is even, $k$ is even fixed as $2$. $Q_{n,k}$ contains four copies of $Q_{n-2,2}$, say $(Q_{n-2,2})_1$, $(Q_{n-2,2})_2$, $(Q_{n-2,2})_3$ and $(Q_{n-2,2})_4$. Let $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, …, $V^i_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, be the included $P_3$-packing $\left\lfloor \frac{n-2}{2} \right\rfloor$- partition sets of $(Q_{n-2,2})_i$, $1 \leq i \leq 4$. One vertex $u_i$, $1 \leq i \leq 4$ in each $(Q_{n-2,2})_i$, $1 \leq i \leq 4$ is not included in any of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, …, $V^i_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, $1 \leq i \leq 4$. For optimal induced $H$-packing $k$-partition, it is necessary that the sub graph induced by $u_1$, $u_2$, $u_3$ and $u_4$ contains a path of length $2$ in $Q_{n,k}$. By the Induction hypothesis, $(Q_{n-2,2})_1, P_3) \geq \left\lfloor \frac{n-2}{2} \right\rfloor$ leaving out one vertex unsaturated. Label the vertices $[V_1^1]$, $[V_2^1]$, $[V_3^1]$,…, $V^1_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, in $(Q_{n-2,2})_1$ as follows.

Let φ be the mapping from $\{[V_1^1], [V_2^1], [V_3^1], \ldots, V^1_{\left\lfloor \frac{n-2}{2} \right\rfloor}]\}$ to $\{1, 2, 3, \ldots, \left\lfloor \frac{n-4}{2} \right\rfloor, \left\lfloor \frac{n-2}{2} \right\rfloor\}$, such that $\varphi([V_a]) = a$.

Similarly $ipp(Q_{n-2,2})_2, P_3)$ is greater than or equal to $\left\lfloor \frac{n-2}{2} \right\rfloor$, each of them leaving out one vertex unsaturated. Label the vertices $[V_1^2]$, $[V_2^2]$, $[V_3^2]$, …, $V^2_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, $(Q_{n-2,2})_2$ as follows.

Let φ be the mapping from $\{[V_1^2], [V_2^2], [V_3^2], \ldots, V^2_{\left\lfloor \frac{n-2}{2} \right\rfloor}]\}$ to $\{1,2,3, \left\lfloor \frac{n-4}{2} \right\rfloor, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor\}$, such that $\varphi([V_a]) = a + 1$. Let $u_1$ be the unsaturated vertex in $(Q_{n-2,2})_1$. Then $|N(u_1)| = n-2$.

By the induction hypothesis the edges incident at vertices of $N(u_1)$ are adjacent to vertices in at most $\left\lfloor \frac{n-4}{2} \right\rfloor$ partition sets. For the same reason $u_i$, $2 \leq i \leq 4$ is adjacent to vertices in at most $\left\lfloor \frac{n-4}{2} \right\rfloor$ partition sets. In $Q_{n,k}$ the unsaturated vertex from each $(Q_{n-2,2})_1$, $(Q_{n-2,2})_2$, $(Q_{n-2,2})_3$ and $(Q_{n-2,2})_4$ induce a vertex disjoint path of length $2$, leaving out one vertex unsaturated. Hence $u_i$, $1 \leq i \leq 4$ is adjacent to at most $\left\lfloor \frac{n-2}{2} \right\rfloor$ partition sets in $Q_{n,k}$. Since $(Q_{n-2,2})_1 \simeq (Q_{n-2,2})_4$ and $(Q_{n-2,2})_2 \simeq (Q_{n-2,2})_3$, the three vertices $u_1$, $u_2$ and $u_3$ are adjacent to at most $\left\lfloor \frac{n-2}{2} \right\rfloor$ partition sets. Therefore $ipp(Q_{n,k}, P_3) \geq \left\lfloor \frac{n}{2} \right\rfloor$.

Suppose n is odd. $Q_{n,k}$ contains two copies of $Q_{n-1,2}$, say $(Q_{n-1,2})_1$, $(Q_{n-1,2})_2$. The induced $P_3$-packing $k$-partition number of $(Q_{n-1,2})_1$ is $\left\lfloor \frac{n-1}{2} \right\rfloor$ leaving out one vertex unsaturated. Since $(Q_{n-1,2})_1$ is even. The role of the partition sets in $(Q_{n-1,2})_1$ is the same as that of $(Q_{n-1,2})_2$. The union is an optimal induced $P_3$-packing $\left\lfloor \frac{n}{2} \right\rfloor$-partition leaving out two unsaturated vertices in $Q_{n,k}$.

**Theorem 1.6.** The induced $P_3$-packing $k$-partition number of $Q_{n,k}$ is $\left\lfloor \frac{n}{2} \right\rfloor$, that is, $ipp(Q_{n,k}, P_3) = \left\lfloor \frac{n}{2} \right\rfloor$, $n \geq 6$.

Proof. Let $[V_1^1]$, $[V_2^1]$, $[V_3^1]$, …, $[V_{\left\lfloor \frac{n-2}{2} \right\rfloor}^1]$, $[V_1^2]$, $[V_2^2]$, $[V_3^2]$, …, $[V_{\left\lfloor \frac{n-2}{2} \right\rfloor}^2]$, $[V_1^3]$, $[V_2^3]$, $[V_3^3]$, …, $[V_{\left\lfloor \frac{n-2}{2} \right\rfloor}^3]$ and $[V_1^4]$, $[V_2^4]$, $[V_3^4]$, …, $[V_{\left\lfloor \frac{n-2}{2} \right\rfloor}^4]$ be the partition sets of $(Q_{n-2,2})_1$, $(Q_{n-2,2})_2$, $(Q_{n-2,2})_3$ and $(Q_{n-2,2})_4$ leaving out one vertex unsaturated respectively. By Lemma 1.4, the binding edges incident at vertices of $[V_i^1]$, $1 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor$, have their other ends in exactly in one $[V_j^2]$, $1 \leq j \leq \left\lfloor \frac{n-2}{2} \right\rfloor$, in $Q_{n-2,k}$.Without loss of generality we say that edges are between $[V_i^1]$, and $[V_{i+1}^1]$, $1 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor$. The role of the partition sets in $(Q_{n-2,2})_1$ is the same as that of $(Q_{n-2,2})_4$ and the partition sets in $(Q_{n-2,2})_2$ is the same as that of $(Q_{n-2,2})_3$. By construction the four left out vertices induce a cycle $C$ in $Q_{n,k}$. Let $P$ be a sub path of length $2$ of $C$ in $Q_{n,k}$. The $\left\lfloor \frac{n-2}{2} \right\rfloor$ partitions sets constructed by our method together with $P$ is an optimal induced $P_3$-packing $\left\lfloor \frac{n}{2} \right\rfloor$ -partition leaving out one vertex unsaturated in $Q_{n,k}$.

**Theorem 1.7**. Let $G$ be the Enhanced hypercube network $Q_{n,2}$, $n \geq 2$. Then $G$ has perfect $C_4$- packing.

Proof. By induction method we prove the result on the dimension $n$ of the Enhanced hypercube network $Q_{n,2}$. We begin with $n = 2$. $P_2 = \{(00, 01, 11, 10)\}$ is a perfect $C_4$-packing. In $(Q_{3,2})$, $P_3 = 0P_2 \cup 1P_2$ is a perfect $C_4$-packing where $iP_2$ denotes the set of paths in $P_2$ prefixed by $i$, $i = 0, 1$.

Assume the result to be true for $Q_{n,2}$. Consider $Q_{n+1,2}$. By induction hypothesis each copy of $Q_{n,2}$ in $Q_{n+1,2}$ contains a perfect $C_4$-packing. The union is a perfect $C_4$-packing in $Q_{n+1,2}$ that is $P_{n+1,2} = 0P_{n,2} \cup 1P_{n,2}$.

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**Lemma 1.8**. *ipp*$(Q_{3,2}, C_4) = 2$.

Proof. Without loss of generality, let $C^1$: $c_1c_2c_3c_4$ be a cycle of length $4$ in $Q_{3,2}$. Then $|\bigcup_{i=1}^{4} N(c_i)| = 4$. Hence another cycle $C^2$ of length $4$ such that $V(C^1) \cap V(C^2) = \varphi$ contains at least one vertex from $|\bigcup_{i=1}^{4} N(c_i)|$. This implies that $ipp(Q_{3,2}, C_4) \geq 2$. Now let $C^1 = \{(010, 000, 001, 011)\}$ and $C^2 = \{(110, 100, 101, 111)\}$. $C^1 \cup C^2$ is an optimal induced $C_4$-packing $2$-partition in $Q_{3,2}$.

**Lemma 1.9.** *ipp*$(Q_{n,2},C_4) = 2$.

Proof. We prove the result by induction on the dimension ($n$, $k$) of the Enhanced hypercube network $Q_{n,2}$. We begin with $n = 5$. $Q_{5,2}$ contains four copies of $Q_{3,2}$, say $(Q_{3,2})_1$, $(Q_{3,2})_2$, $(Q_{3,2})_3$, $(Q_{3,2})_4$. Let $[V_i^1]$ and $[V_i^2]$ be the induced $2$-partition sets of $(Q_{4,2})_i$, $i = 1,2,3,4$. The binding edges incident at vertices of $[V_i^1]$, $1 \leq i \leq 2$, have their other ends in exactly in one $[V_j^2]$, $1 \leq j \leq 2$ in $Q_{5,3}$. By Lemma 1.8, $ipp(Q_{3,2})C_4)$ is $2$. Let $[V_1]$, $[V_2]$ be the induced $C_4$-packing $2$-partition sets of $(Q_{3,2})_1$. Without loss of generality each of $[V_1^1]$, $[V_2^1]$, contains at most $4$ vertices of $V(Q_{3,2})_1$. Let$[V_1] = \{C^1\}$, where $C^1$: $a_1a_2a_3a_4$ is in $(Q_{3,2})_1$. Then $|\bigcup_{i=1}^{4} N(a_i) \cap (Q_{3,2})_2| = 4$.

Hence $\bigcup_{i=1}^{4} N(a_i) \cap (Q_{3,2})_2$ is not in $[V_1]$. This implies $\bigcup_{i=1}^{4} N(a_i) \cap (Q_{3,2})_2$ is in $[V_2]$. Let $[V_2] = \{C^2\}$, where $C^2$: $b_1b_2b_3b_4$ is in $(Q_{3,2})_1$. Then$|\bigcup_{i=1}^{4} N(b_i) \cap (Q_{3,2})_2| = 4$. Hence $\bigcup_{i=1}^{4} N(b_i) \cap (Q_{3,2})_2$ is not in $[V_2]$. This implies $\bigcup_{i=1}^{4} N(b_i) \cap (Q_{3,2})_2$ is in $[V_1]$. Similarly $(Q_{3,2})_3$ is partitioned as in $(Q_{3,2})_2$ and $(Q_{3,2})_4$ is partitioned as in $(Q_{3,2})_1$. $[V_1^i] \cup [V_2^i]$ $i = 1, 2, 3, 4$ is an optimal induced $C_4$-packing $2$-partition in $Q_{5,2}$. Assume that the result is true for $Q_{n-1,2}$. $Q_{n,2}$ contains two copies of $Q_{n-1,2}$, say $(Q_{n-1,2})_1$ and $(Q_{n-1,2})_2$. By the induction hypothesis $ipp((Q_{n-1,2})_1, C_4)$ is $2$. Since $(Q_{n-1,2})_1 \simeq (Q_{n-1,2})_2$, $[V_1^i] \cup [V_2^i]$, $i = 1, 2$ is an optimal induced $C_4$-packing $2$-partition in $Q_{n,2}$.

## III. AUGMENTED CUBES

We define the augmented cube $AQ_n$. As with hypercubes, augmented cubes admit several definitions.

Let $n \geq 1$ be an integer. The Augmented cube $AQ_n$ of dimension $n$ has $2^n$ vertices each labelled by an $n$ bit binary string $a_1a_2a_3..... a_n$. We define $AQ_1 = K_2$. For $n \geq 2$, $AQ_n$ is obtained by taking two copies of the augmented cube $AQ_{n-1}$, denoted by $AQ^0_{n-1}$ and $AQ^1_{n-1}$, and adding $2*2^{n-1}$ edges between the two as follows [5], [6]. Let $V(AQ^0_{n-1}) = 0a_2a_3....a_n$; $a_i = 0$ or $1$ and $V(AQ^1_{n-1}) = 0b_2b_3....b_n$; $b_i=0$ or $1$. A vertex $A = 0a_2a_3....a_n$ of $(AQ^0_{n-1})$ is joined to a vertex $B = 1b_2b_3....b_n$ of $(AQ^1_{n-1})$ if and only if for every $i$, $2 \leq i \leq n$ either

(i)$a_i$=$b_i$; in this case, $AB$ is called hypercube edge, or

(ii)$a_i = b_i$; in this case, $AB$ is called complementary edge.

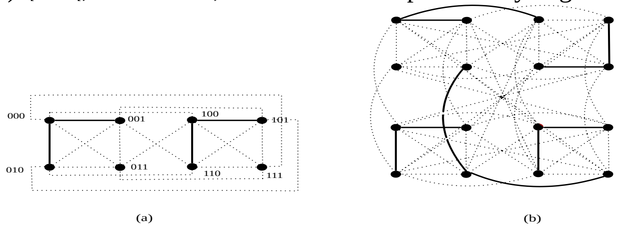


**Fig. 3. (*a*) An induced $P_3$-packing *2*-partition number of $AQ_3$ (*b*) An induced $P_3$-packing *3* - partition number of $AQ_4$**

**Lemma 2.1.** *ipp*$(AQ_4, P_3) = 5$.

Proof. $AQ_4$ is packed with $5$ vertex disjoint paths of length $2$, leaving out one vertex unsaturated. Suppose $ipp(AQ_4, P_3) = 4$. Let $[V_1]$, $[V_2]$, $[V_3]$ and $[V_4]$ be the induced $P_3$-packing $4$-partition sets. The possibility is, (i) $\|[V_1]\| = 2$, $\|[V_2]\| = 1$, $\|[V_3]\| = 1$ and $\|[V_4]\| = 1$. We claim that $\|[V_1]\| \geq 2$ is not possible.

Suppose $\|[V_1]\|=2$. Let $P$: $uvw$ be in $[V_1]$. Then $|N(u) \cup N(v) \cup N(w)|=9$. Now $V(P)$ and its neighboring vertices constitute $12$ vertices leaving $4$ vertices unsaturated. If $Q$ is the other path of length $2$ in $[V_1]$, then $N(V(P))=N(V(Q))$, a contradiction.

If $\|[V_1]\| = 2$ is not possible, then $\|[V_1]\| \geq 2$ . The only possible way is $\|[V_1]\| = 1$, $\|[V_2]\| = 1$, $\|[V_3]\| = 1$, $\|[V_4]\| = 1$, $\|[V_5]\| = 1$.

This implies that $ipp(AQ_4) \geq 5$. Now let $P = \{(0001, 0000, 0100)\}$, $Q = \{(0101, 0111, 0110)\}$, $R = \{(1001, 1000, 1010)\}$, $S = \{(1101, 1100, 1110)\}$, $T = \{(1111, 1011, 0011)\}$. $P \cup Q \cup R \cup S \cup T$ is an optimal induced $P_3$-packing $5$-partition leaving out one vertex unsaturated in $AQ_4$. See Fig. 3(b).

**Lemma 2.2.** *ipp*$(AQ_6, P_3) \geq 5$.

Proof. $AQ_6$ contains four copies of $AQ_4$, say $(AQ_4)_i$, $1 \leq i \leq 4$. By Lemma 2.1, $ipp(AQ_4) \geq 5$. Let $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $[V_4^i]$, $[V_5^i]$ be the induced $P_3$-packing $5$-partition sets of $(AQ_4)_i$, $1 \leq i \leq 4$. One vertex $u_i$, $1 \leq i \leq 4$ in each $(AQ_4)_i$, $1 \leq i \leq 4$ is not included in any of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $[V_4^i]$, $[V_5^i]$, $1 \leq i \leq 4$. For optimal induced $P_3$-packing $5$-partition, it is necessary that the sub graph induced by $u_1$, $u_2$, $u_3$ and $u_4$ contains a path of length $2$ in $AQ_6$. Consider $u_1$ in $(AQ_4)_1$, $deg(AQ_4)_1\ (u_1)=7$. If $u_1$ is adjacent to vertices in $[V_1^1]$, $[V_2^1]$ ,$[V_3^1]$, $[V_4^1]$ and $[V_5^1]$ then the $P_3$-path containing $u_1$ cannot be included in any of $[V_1^1]$, $[V_2^1]$, $[V_3^1]$, $[V_4^1]$or$[V_5^1]$ a contradiction. Suppose one vertex adjacent to $u_1$ is in $[V_5^1]$, two vertices adjacent to $u_1$ are in $[V_1^1]$, two vertices adjacent to $u_1$ are in $[V_2^1]$, one vertex adjacent to $u_1$ is $[V_3^1]$, one vertex adjacent to $u_1$ $[V_4^1]$ then a $5$-cycle is induced by these vertices, a contradiction. For the same reason, $u_1$ cannot be adjacent to $7$ vertices in $[V_i^1]$ with $\|[V_i^1]\| = 2$, $1 \leq i \leq 5$. Hence $u_1$ is adjacent to $7$ vertices in any one of $[V_i^1]$ with $\|[V_i^1]\| = 1$, $1 \leq i \leq 5$. This argument is true for $u_i$ in $(AQ_4)_i$ , $2 \leq i \leq 4$. We now claim that the binding edges in $(AQ_4)_1 \cup (AQ_4)_2) \setminus (AQ_4)_1$ incident at vertices of $[V_i^1]$, $1 \leq i \leq 5$, have their other ends in exactly one $[V_j^2]$ , $1 \leq j \leq 5$. Suppose not, without loss of generality let all the end vertices of binding edges incident at vertices of $[V_1^1]$ be adjacent to vertices in $[V_2^2]$, also end vertices of binding edges incident at vertices of $[V_2^1]$

be adjacent to vertices in $[V_1^2]$ and end vertices of binding edges incident at vertices of $[V_3^1]$ be adjacent to vertices in $[V_4^2]$ also end vertices of binding edges incident at vertices of $[V_4^1]$ be adjacent to vertices in $[V_5^2]$ also end vertices of binding edges incident at vertices of $[V_5^1]$ be adjacent to vertices in $[V_3^2]$, then no vertex in $[V_5^1]$ is adjacent to any vertex in $[V_3^2]$, a contradiction. This argument is also true for $[V_i^1]$, i = $2,3,4,5$. Now $V(AQ_4)$ can be partitioned into $[V_1]$, $[V_2]$, $[V_3]$, $[V_4]$ and $[V_5]$ such that, each of $[V_1]$, $[V_2]$, $[V_3]$, $[V_4]$ and $[V_5]$ contains at most $3$ vertices of $V(AQ_4)$.Therefore we have $\|[V_1]\| =1$, $\|[V_2]\| = 1$, $\|[V_3]\| = 1$, $\|[V_4]\| = 1$ and $\|[V_5]\| = 1$. Let $[V_1] =\{P\}$, where $P$: $p_1p_2p_3$ are in $(AQ_4)_1$. Then $|\bigcup_{i=1}^{3} N(p_i) \cap (AQ_4)_2| = 3$. Hence $\bigcup_{i=1}^{3} N(p_i) \cap (AQ_4)_2$ are not in $[V_1]$.This implies $\bigcup_{i=1}^{3} N(p_i) \cap (AQ_4)_2$ are in $[V_2]$. Now let $[V_2] = \{Q\}$, where $Q$:$q_1q_2q_3$ are in $(AQ_4)_1$.Then

$|\cup_{i=1}^{3} N(q_i) \cap (AQ_4)_2| = 3$. Hence $\cup_{i=1}^{3} N(q_i) \cap (AQ_4)_2$ are not in $[V_2]$. This implies $\cup_{i=1}^{3} N(q_i) \cap (AQ_4)_2$ are in $[V_1]$. Let $[V_3] = \{R\}$, where $R: r_1r_2r_3$ is in $(AQ_4)_1$. Then $|\cup_{i=1}^{3} N(r_i) \cap (AQ_4)_2| = 3$. Hence $\cup_{i=1}^{3} N(r_i) \cap (AQ_4)_2$ is not in $[V_3]$. This implies $\cup_{i=1}^{3} N(r_i) \cap (AQ_4)_2$ is in $[V_4]$. Let $[V_4] = \{S\}$, where $S: s_1s_2s_3$ is in $(AQ_4)_1$. Then $|\cup_{i=1}^{3} N\, s_i) \cap (AQ_4)_2| = 3$. Hence $\cup_{i=1}^{3} N(s_i) \cap (AQ_4)_2$ is not in $[V_4]$. This implies $\cup_{i=1}^{3} N(s_i) \cap (AQ_4)_2$ is in $[V_5]$. Let $[V_5] = \{T\}$ where $T: t_1t_2t_3$ is in $(AQ_4)_1$. Then $|\cup_{i=1}^{3} N(t_i) \cap (AQ_4)_2| = 3$. Hence $\cup_{i=1}^{3} N(t_i) \cap (AQ_4)_2$ is not in $[V_5]$. This implies $\cup_{i=1}^{3} N(t_i) \cap (AQ_4)_2$ is in $[V_3]$.

Similarly $(AQ_4)_3$ is partitioned as in $(AQ_4)_2$ and $(AQ_4)_4$ is partitioned as in $(AQ_4)_1$. Let $u_1$ be the unsaturated vertex in $(AQ_1)_4$. Then $|N(u_1)|=7$.

Hence the edges incident at vertices of $N(u_1)$ are adjacent to vertices in any one of $[V_i^1]$ with $|[V_i^1]| = 1$, $1 \le i \le 5$.

Without loss of generality let $u_1$ be adjacent to a vertex in $[V_1^1]$. Similarly let $u_2$ be the unsaturated vertex in $(AQ_4)_2$. Since $|N(u_2)|=7$, the edges incident at vertices of $N(u_2)$ are adjacent to vertices in any one of $[V_i^2]$ with $|[V_i^2]|=1$, $1 \le i \le 5$. This implies $u_2$ is adjacent to a vertex in $[V_1^2]$. For the same reason $u_3$ is adjacent to a vertex in $[V_1^3]$ and $u_4$ is adjacent to a vertex in $|[V_1^4]|$ Hence the edges incident at vertex $u_i$, $1 \le i \le 5$ are adjacent to vertices in at most one of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $[V_4^i]$, $[V_5^i]$ $1 \le i \le 4$ in each $(AQ_4)_i$, $1 \le i \le 4$. This implies $u_i$, $1 \le i \le 4$ is adjacent to at most one of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $[V_4^i]$, $[V_5^i]$, $1 \le i \le 5$ in $AQ_6$. Since $(AQ_4)_1 \simeq (AQ_4)_4$ and $(AQ_4)_2 \simeq (AQ_4)_3$, the unsaturated vertices from each $(AQ_4)_1$, $(AQ_4)_2$, $(AQ_4)_3$ and $(AQ_4)_4$ induce a vertex disjoint path of length $2$, leaving out one vertex unsaturated. This implies that the three vertices $u_1$, $u_2$ and $u_3$ are adjacent to at most four vertex sets. Therefore $ipp(AQ_6) \ge 5$.

**Lemma 2.3**. $ipp(AQ_n) \ge \left\lfloor \frac{n}{2} \right\rfloor + 2$, $n \ge 6$.

Proof. By induction method, we prove the result on the dimension $n$ of the Augmented cube network $AQ_n$. We prove something more and prove that an unsaturated vertex $u_i$, $1 \le i \le 4$ in $(AQ_{n-2})_i$, $1 \le i \le 4$ is adjacent to $\left\lfloor \frac{n-4}{2} \right\rfloor$ vertices in $\left\lfloor \frac{n-4}{2} \right\rfloor$ partition sets of $(AQ_{n-2})_i$, $1 \le i \le 4$. We begin with $n=8$. $AQ_8$ contains four copies of $AQ_6$, say $(AQ_6)_i$, $1 \le i \le 4$. By lemma, 2.2 $ippAQ_6 \ge \left\lfloor \frac{n}{2} \right\rfloor + 2$, leaving out one vertex unsaturated. Let $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $[V_4^i]$ and $[V_5^i]$ be the induced $P_3$-packing $5$-partition sets of $(AQ_6)_i$, $1 \le i \le 4$. One vertex $u_i$, $1 \le i \le 4$ in each $(AQ_6)_i$, $1 \le i \le 4$ is not included in any of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $[V_4^i]$, $[V_5^i]$, $1 \le i \le 4$. For optimal induced $P_3$-packing $\left\lfloor \frac{n}{2} \right\rfloor + 2$ partition, it is necessary that the sub graph induced by $u_1$, $u_2$, $u_3$ and $u_4$ contains a path of length $2$ in $AQ_8$. Consider $u_1$ in $(AQ_6)_1$, $deg(AQ_6)_1(u_1) = 11$. If $u_1$ is adjacent to vertices in $[V_1^1]$, $[V_2^1]$, $[V_3^1]$, $[V_4^1]$ and $[V_5^1]$ then the $5$-path containing $u_1$ cannot be included in any of $[V_1^1]$, $[V_2^1]$, $[V_3^1]$, $[V_4^1]$ and $[V_5^1]$ a contradiction. Suppose $u_1$ is adjacent to vertices in any one of $[V_i^1]$, $1 \le i \le 5$, then $ipp(AQ_6) > \left\lfloor \frac{n}{2} \right\rfloor + 2$ a contradiction. Hence $u_1$ is adjacent to $11$ vertices in at most one of $[V_i^1]$, $1 \le i \le 5$. This argument is also true for $u_i$ in $(AQ_6)_i$, $2 \le i \le 4$. This implies that the three vertices $u_1$, $u_2$ and $u_3$ are adjacent to at most five vertex sets. This implies $ipp(AQ_8) \ge \left\lfloor \frac{n}{2} \right\rfloor + 2$.

Assume the result is true for Augmented cube with dimension less than or equal to $n-1$. Consider $AQ_n$. When $n$ is even. $AQ_n$ contains four copies of $AQ_{n-2}$, say $(AQ_{n-2})_1$, $(AQ_{n-2})_2$, $(AQ_{n-2})_3$ and $(AQ_{n-2})_4$. Let $[V_1^i]$, $[V_2^i]$, $[V_3^i]$,…, $V^i_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, be the included $P_3$-packing $\left\lfloor \frac{n}{2} \right\rfloor + 2$ partition sets of $(AQ_{n-2})_i$, $1 \le i \le 4$.

One vertex $u_i$, $1 \le i \le 4$ in each $(AQ_{n-2})_i$, $1 \le i \le 4$ is not included in any of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, …, $V^i_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, $1 \le i \le 4$. For optimal induced $H$-packing $k$-partition, it is necessary that the sub graph induced by $u_1$, $u_2$, $u_3$ and $u_4$ contains a path of length $2$ in $AQ_n$. By the induction hypothesis, $ipp(AQ_{n-2})_1) \ge \left\lfloor \frac{n}{2} \right\rfloor + 2$ leaving out one vertex unsaturated. Label the vertices of $[V_1^1]$, $[V_2^1]$, $[V_3^1]$,…, $V^1_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, in $(AQ_{n-2})_1$. Let $\varphi$ be the mapping from $\{[V_1^1]$, $[V_2^1]$, $[V_3^1]$,…, $V^1_{\left\lfloor \frac{n-4}{2} \right\rfloor}]$, $V^1_{\left\lfloor \frac{n-2}{2} \right\rfloor}]\}$, to $\{1,2,3,\ldots, \left\lfloor \frac{n-4}{2} \right\rfloor, \left\lfloor \frac{n-2}{2} \right\rfloor\}$, such that $\varphi([V_a]) = a$. Similarly $ipp(AQ_{n-2})_2$ is greater than or equal to $\left\lfloor \frac{n-2}{2} \right\rfloor$, each of them leaving out one vertex unsaturated. Label the vertices of $[V_1^2]$, $[V_2^2]$, $[V_3^2]$,…, $V^2_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$ in $(AQ_{n-2})_2$ as follows. Let $\varphi$ be the mapping from $\{[V_1^2]$, $[V_2^2]$, $[V_3^2]$, …, $V^2_{\left\lfloor \frac{n-4}{2} \right\rfloor}]$, $V^2_{\left\lfloor \frac{n-2}{2} \right\rfloor}]\}$, to $\{1, 2, 3,\ldots, \left\lfloor \frac{n-4}{2} \right\rfloor, \left\lfloor \frac{n-2}{2} \right\rfloor\}$ such that $\varphi([V_a]) = a + 1$.

Let $u_1$ be the unsaturated vertex in $(AQ_{n-2})_1$. Then $|N(u_1)| = n-2$. By the induction hypothesis the edges incident at vertices of $N(u_1)$ are adjacent to vertices in at most $\left\lfloor \frac{n-4}{2} \right\rfloor$ partition sets. For the same reason $u_i$, $2 \le i \le 4$ is adjacent to vertices in at most $\left\lfloor \frac{n-4}{2} \right\rfloor$ partition sets. In $AQ_n$ the unsaturated vertex from each $(AQ_{n-2})_1$, $(AQ_{n-2})_2$, $(AQ_{n-2})_3$ and $(AQ_{n-2})_4$ induce a vertex disjoint path of length $2$, leaving out one vertex unsaturated. Hence $u_i$, $1 \le i \le 4$ is adjacent to at most $\left\lfloor \frac{n-2}{2} \right\rfloor + 2$ partition sets in $AQ_n$. Since $(AQ_{n-2})_1 \simeq (AQ_{n-2})_4$ and $(AQ_{n-2})_2 \simeq (AQ_{n-2})_3$, the three vertices $u_1$, $u_2$ and $u_3$ are adjacent to at most $\left\lfloor \frac{n-2}{2} \right\rfloor + 2$ partition sets. Therefore $ipp(AQ_n) \ge \left\lfloor \frac{n}{2} \right\rfloor + 2$. Suppose $n$ is odd. $AQ_n$ contains two copies of $AQ_{n-1}$, say $(AQ_{n-1})_1$ and $(AQ_{n-1})_2$. The induced $P_3$-packing $k$-partition number of $(AQ_{n-1})_1$ is $\left\lfloor \frac{n-1}{2} \right\rfloor$ leaving out one vertex unsaturated. Since $(AQ_{n-1})_1$ is even. The role of the partition sets in $(AQ_{n-1})_1$ is the same as that of $(AQ_{n-1})_2$.

The union is an optimal induced $P_3$-packing $\left\lfloor \frac{n}{2} \right\rfloor + 2$ partition leaving out two unsaturated vertices in $AQ_n$.

**Theorem 2.4**. $ipp(AQ_n) = \left\lfloor \frac{n}{2} \right\rfloor + 2$, $n \ge 6$.

Proof. $[V_1^1]$, $[V_2^1]$, $[V_3^1]$,…, $[V^1_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, $[V_1^2]$, $[V_2^2]$,$[V_3^2]$,…, $[V^2_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, $[V_1^3]$, $[V_2^3]$, $[V_3^3]$,…, $[V^3_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$ and $[V_1^4]$, $[V_2^4]$, $[V_3^4]$, …, $[V^4_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$ be the partition sets $(AQ_{n-2})_1$, $(AQ_{n-2})_2$, $(AQ_{n-2})_3$ and $(AQ_{n-2})_4$ leaving out one vertex unsaturated respectively.

By Lemma 2.2, the binding edges incident at vertices of $[V_i^1]$, $1 \le i \le \left\lfloor \frac{n-2}{2} \right\rfloor$, have their other ends in exactly in one $[V_j^2]$, $1 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor$, in $AQ_{n-2}$. Without loss of generality we say that edges are between $[V_i^1]$ and $[V_{i+1}^1]$, $1 \le i \le \left\lfloor \frac{n-2}{2} \right\rfloor$. The role of the partition sets in $(AQ_{n-2})_1$ is the same as that of $(AQ_{n-2})_4$ and the partition sets in $(AQ_{n-2})_2$ is the same as that of

$(AQ_{n-2})_3$. By construction the four left out vertices induce a cycle $C$ in $AQ_n$. Let $P$ be a sub path of length $2$ of $C$ in $AQ_n$. The $\left\lfloor \frac{n-2}{2} \right\rfloor +2$ partitions sets constructed by our method together with $P$ is an optimal induced $P_3$-packing $\left\lfloor \frac{n}{2} \right\rfloor +2$ partition leaving out one vertex unsaturated in $AQ_n$.

**Lemma 2.5.** The Augmented Cube $AQ_n$, $n \geq 2$, Then $AQ_n$ has perfect $C_4$-packing.

Proof. Follows from Lemma 1.7.

**Lemma 2.6.** The induced $C_4$-packing $k$-partition number Augmented $AQ_3$ is $2$, that is, $ipp(AQ_3, C_4) = 2$.

Proof. Follows from Lemma 1.8.

**Lemma 2.7.** $ipp(AQ_n,C_4) = 4$.

Proof. We prove the result by induction on the dimension $n$ of the Augmented cube network $AQ_n$. We begin with $n = 5$. $AQ_5$ contains four copies of $AQ_3$, say $(AQ_3)_1$, $(AQ_3)_2$, $(AQ_3)_3$, $(AQ_3)_4$. Let $[V_1]$, $[V_2]$, $[V_3]$, $[V_4]$ be the induced $C_4$-packing $4$-partition sets of $(AQ_3)_i$, $i = 1, 2, 3, 4$. By lemma 2.6, $ipp(AQ_3, C_4)$ is $2$. In $(AQ_3)_1$ each of $[V_1^1]$, $[V_2^1]$, contains at most $4$ vertices of $V(AQ_3)_1)$. Let $[V_1] =\{C^1\}$, where $C^1$: $a_1a_2a_3a_4$ is in $(AQ_3)_1$. $|\bigcup_{i=1}^{4} N(a_i) \cap (AQ_3)_2| = 4$. Hence $\bigcup_{i=1}^{4} N(a_i) \cap (AQ_3)_2$ is not in $[V_1]$. This implies $\bigcup_{i=1}^{4} N(a_i) \cap (AQ_3)_2$ is in $[V_2]$. Let $[V_2] = \{C^2\}$, where $C^2$: $b_1b_2b_3b_4$ is in $(AQ_3)_1$. Then $|\bigcup_{i=1}^{4} N(b_i) \cap (AQ_3)_2| = 4$. Hence $\bigcup_{i=1}^{4} N(b_i) \cap (AQ_3)_2$ is not in $[V_2]$. This implies $\bigcup_{i=1}^{4} N(b_i) \cap (AQ_3)_2$ is in $[V_1]$. Let $[V_3]$, $[V_4]$ be the induced $C_4$-packing partition sets of $(AQ_3)_3$. Each of $[V_3^3]$, $[V_4^3]$ contains at most $4$ vertices of $V(AQ_3)_3$. Let $[V_3] =\{C^3\}$, where $C^3$: $c_1c_2c_3c_4$ is in $(AQ_3)_3$. Then $|\bigcup_{i=1}^{4} N(c_i) \cap (AQ_3)_4| = 4$. Hence $\bigcup_{i=1}^{4} N(c_i) \cap (AQ_3)_4$ is not in $[V_3]$. This implies $\bigcup_{i=1}^{4} N(c_i) \cap (AQ_3)_4$ is in $[V_4]$. Let $[V4] = \{C^4\}$,where $C^4$:$d_1d_2d_3d_4$ is in $(AQ_3)_4$. Then Then $|\bigcup_{i=1}^{4} N(d_i) \cap (AQ_3)_4| = 4$. Hence $\bigcup_{i=1}^{4} N(d_i) \cap (AQ_3)_4$ is not in $[V_4]$. This implies $\bigcup_{i=1}^{4} N(d_i) \cap (AQ_3)_4$ is in $[V_3]$. $[V_1^i] \cup [V_2^i] \cup V_3^i] \cup [V_4^i]$, $i =1, 2, 3, 4$ is an optimal induced $C_4$-packing $4$-partition in $AQ_5$. Assume that the result is true for $AQ_n$. When n is odd. $AQ_n$ contains two copies of $AQ_{n-1}$, say $(AQ_{n-1})_1$ and $(AQ_{n-1})_2$. By the induction hypothesis $ipp(AQ_{n-1})_1,C_4)$ is $4$.

## IV. CROSSED CUBE NETWORKS

The crossed cube has additional attractive properties. It has more cycles than the hypercube. A crossed cube of $n$ dimensions, denoted by $CQ_n$, has $2^n$ vertices. Each vertex of $CQ_n$ is identified by a unique $n$-bit binary string; e.g. vertex $u = u_nu_{n-1},..., u_2u_1$, where $u_i$ $0, 1$ for $1 \leq i \leq n$. The following are the formal definitions. Two binary strings $x = x_2x_1$ and $y = y_2y_1$ of length two are pair related, denoted by $x$ $y$ if and only if $(x, y) = (00, 00)$, $(10, 10)$, $(01, 11)$, $(11, 01)$ [1] [3]. The $n$- dimensional crossed cube $(CQ_n)$ is a $n$ - label graph, it can be defined as follows. $CQ_1$ is

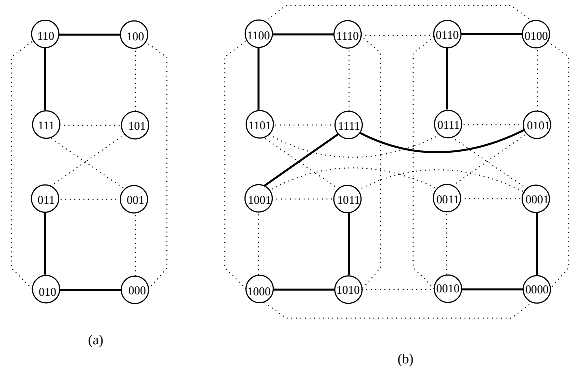


Fig. 4. (*a*) An induced $P_3$-packing $2$-partition number of $CQ_3$ (*b*) An induced $P_3$-packing $3$ - partition number of $CQ_4$.

$k_2$, the complete graph of two vertices with labels $0$ and $1$; for $n>1$, $(CQ_n)$ consists of two $(n-1)$ dimensional crossed cube $CQ_{n-1}^0$ and $CQ_{n-1}^1$, where $V(CQ_{n-1}^i) = x_nx_{n-1}.....x_1/x_n = i$, $(i = 0, 1)$.The vertex $x = 0x_{n-1}\,x_{n-2}...x_1$ in $CQ_{n-1}^0$ and the vertex $y = 1y_{n-1}\,y_{n-2}....y_1$ in $CQ_{n-1}^1$ are adjacent in $CQ_n$ if:

(1) $x_{n-1} = y_{n-1}$ if $n$ is even,

(2) For $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, $x_{2i}x_{2i-1} \sim y_{2i}y_{2i-1}$ [1].

**Theorem 3.1.** Let $G$ be the Crossed cube network $CQ_n(n \geq 2)$. Then G has an almost perfect $P_3$-packing.

Proof. Follows from Theorem 1.1.

**Lemma 3.2.** $ipp(CQ_3, P_3) = 2$.

Proof. Follows from Lemma 1.2.

**Lemma 3.3.** $ipp(CQ_4, P_3) = 3$.

Proof. Follows from Lemma 1.3.

**Lemma 3.4.** $ipp(CQ_6, P_3) \geq 3$.

Proof. $CQ_6$ contains four copies of $CQ_4$, say $(CQ_4)_i$, $1 \leq i \leq 4$. By Lemma 3.3, $ipp(CQ_4) \geq 3$. Let $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $[V_4^i]$ be the induced $P_3$-packing $4$-partition sets of $(CQ_4)_i$, $1 \leq i \leq 4$. One vertex $u_i$, $1 \leq i \leq 4$ in each $(CQ_4)_i$, $1 \leq i \leq 4$ is not included in any of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, $[V_4^i]$, $1 \leq i \leq 4$. For optimal induced $P_3$-packing $4$-partition, it is necessary that the sub graph induced by $u_1$, $u_2$, $u_3$ and $u_4$ contains a path of length $2$ in $CQ_6$. Consider $u_1$ in $(CQ_4)_1$, $deg(CQ_4)_1\,(u_1) = 4$. If $u_1$ is adjacent to vertices in $[V_1^1]$, $[V_2^1]$, $[V_3^1]$ and $[V_4^1]$ then the $P_3$-path containing $u_1$ cannot be included in any of $[V_1^1]$, $[V_2^1]$, $[V_3^1]$ or $[V_4^1]$ a contradiction. Suppose two vertices adjacent to $u_1$ are in $[V_1^1]$ and one vertex adjacent to $u_1$ is in $[V_2^1]$ and one vertex adjacent to $u_1$ is in $[V_3^1]$ then a $3$-cycle is induced by these vertices, a contradiction. For the same reason, $u_1$ cannot be adjacent to $4$ vertices in $[V_i^1]$ with $|[V_i^1]| = 1$, $1 \leq i \leq 4$. Hence $u_1$ is adjacent to $4$ vertices in any one of $[V_i^1]$ with $|[V_i^1]| = 2$, $1 \leq i \leq 4$. This argument is also true for $u_i$ in $(CQ_4)_i$, $2 \leq i \leq 4$. We claim that the binding edges in $(CQ_4)_1 \cup (CQ_4)_2) \backslash (CQ_4)_1$ incident at vertices of $[V_i^1]$, $1 \leq i \leq 4$, have their other ends in exactly one $[V_j^2]$, $1 \leq j \leq 4$. Suppose not, without loss of generality, let all the end vertices of binding edges incident at vertices of $[V_1^1]$ be adjacent to vertices in $[V_3^2]$ and $[V_4^2]$ also end vertices of binding edges incident at vertices of $[V_2^1]$ be adjacent to vertices in $[V_3^2]$ and $[V_4^2]$ and end vertices of binding edges incident at vertices of $[V_3^1]$ be adjacent to vertices in $[V_2^2]$ $[V_1^2]$ $[V_4^2]$ end

vertices of binding edges incident at vertices of $[V_4^1]$ be adjacent to vertices in $[V_3^2]$ and $[V_2^2]$ $[V_1^2]$, then no vertex in $[V_4^1]$ is adjacent to any vertex in $[V_1^2]$, $[V_2^2]$ and $[V_3^2]$ a contradiction. This argument is also true for $[V_i^1], i = 2, 3, 4$. Now $V(CQ_4)$ can be partitioned into $[V_1]$, $[V_2]$, $[V_3]$ and $[V_4]$ such that, each of $[V_1]$, $[V_2]$, $[V_3]$ contains at most $9$ vertices of $V(CQ_4)$ and $[V_4]$ contains at most $6$ vertices of $V(CQ_4)$. We have $|[V_1]| = 1$, $|[V_2]| = 1$ and $|[V_3]| = 1$, $|[V_4]| = 2$. Let $[V_1] = \{P\}$, where $P$: $p_1p_2p_3$ are in $(CQ_4)_1$. Then $|\bigcup_{i=1}^{3} N(p_i) \cap (CQ_4)_2| = 3$.
Hence $\bigcup_{i=1}^{3} N(p_i) \cap (CQ_4)_2$ is not in $[V_1]$. This implies $\bigcup_{i=1}^{3} N(p_i) \cap (CQ_4)_2$ is in $[V_3]$ and $[V_4]$. Now let $[V_2] = \{Q\}$, where $Q$: $q_1q_2q_3$ in $(CQ_4)_1$. Then $|\bigcup_{i=1}^{3} N(q_i) \cap (CQ_4)_2| = 3$. Hence $\bigcup_{i=1}^{3} N(q_i) \cap (CQ_4)_2$ is not in $[V_2]$. This implies $\bigcup_{i=1}^{3} N(q_i) \cap (CQ_4)_2$ is in $[V_3]$ and $[V_4]$. Let $[V_3] = \{R\}$, where $R$: $r_1r_2r_3$ are in $(CQ_4)_1$. Then $|\bigcup_{i=1}^{3} N(r_i) \cap (CQ_4)_2| = 3$. Hence $\bigcup_{i=1}^{3} N(r_i) \cap (CQ_4)_2$ is not in $[V_3]$. This implies $\bigcup_{i=1}^{3} N(r_i) \cap (CQ_4)_2$ is in $[V_1]$ and $[V_2]$. Let $[V_4] = \{S, T\}$, where $s_1s_2s_3$ and $t_1t_2t_3$ are in $(CQ_4)_1$. Then $|\bigcup_{i=1}^{3} N(s_i) \cap (CQ_4)_2| = 3$ and $|\bigcup_{i=1}^{3} N(t_i) \cap (CQ_4)_2| = 3$. Hence $\bigcup_{i=1}^{3} N(s_i) \cap (CQ_4)_2$ and $\bigcup_{i=1}^{3} N(t_i) \cap (CQ_4)_2$ are not in $[V_4]$. This implies $\bigcup_{i=1}^{3} N(s_i) \cap (CQ_4)_2$ and $\bigcup_{i=1}^{3} N(t_i) \cap (CQ_4)_2$ are in $[V_2]$, $[V_3]$ and $[V_1]$.
Similarly $(CQ_4)_3$ is partitioned as in $(CQ_4)_2$ and $(CQ_4)_4$ is partitioned as in $(CQ_4)_1$. Let $u_1$ be the unsaturated vertex in $(CQ_4)_1$. Then $|N(u_1)| = 4$. Hence the edges incident at vertices of $N(u_1)$ are adjacent to vertices in any one of $[V_i^1]$ with $|[V_i^1]| = 1$, $1 \le i \le 4$. Without loss of generality, let $u_1$ be adjacent to a vertex in $[V_1^1]$. Similarly let $u_2$ be the unsaturated vertex in $(CQ_4)_2$. Since $|N(u_2)| = 4$, the edges incident at vertices of $N(u_2)$ are adjacent to vertices in any one of $[V_i^2]$ with $|[V_i^2]| = 1$, $1 \le i \le 4$. This implies $u_2$ is adjacent to a vertex in $[V_1^2]$ For the same reason $u_3$ is adjacent to a vertex in $[V_1^3]$ and $u_4$ is adjacent to a vertex in $[V_1^4]$. Hence the edges incident at vertex $u_i$, $1 \le i \le 4$ are adjacent to vertices in at most one of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$ and $[V_4^i]$, $1 \le i \le 4$ in each $(CQ_4)_i$, $1 \le i \le 4$. This implies $u_i$, $1 \le i \le 4$ is adjacent to at most two of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$ and $[V_4^i]$, $1 \le i \le 4$ in $CQ_6$. Since $(CQ_4)_1 \simeq (CQ_4)_4$ and $(CQ_4)_2 \simeq (CQ_4)_3$, the unsaturated vertices from each $(CQ_4)_1$, $(CQ_4)_2$, $(CQ_4)_3$ and $(CQ_4)_4$ induce a vertex disjoint path of length $2$, leaving out one vertex unsaturated. This implies that the three vertices $u_1$, $u_2$ and $u_3$ are adjacent to at most three vertex sets. Therefore $ipp(CQ_6) = 4$.

**Lemma 3.5.** $ipp(CQ_n, P_3) \ge \left\lfloor \frac{n}{2} \right\rfloor + 1, n \ge 6$.

Proof. Using induction method, we prove the result on the dimension $n$ of the Crossed cube network $CQ_n$. We prove something more and prove that an unsaturated vertex $u_i$, $1 \le i \le 4$ in $(CQ_{n-2})_i$, $1 \le i \le 4$ is adjacent to $\left\lfloor \frac{n-4}{2} \right\rfloor$ vertices in $\left\lfloor \frac{n-4}{2} \right\rfloor$ partition sets of $(CQ_{n-2})_i$, $1 \le i \le 4$. We begin with $n = 8$. $CQ_8$ contains four copies of $CQ_6$, say $(CQ_6)_i$, $1 \le i \le 4$. By lemma (3.4), $ipp(CQ_6) \ge \left\lfloor \frac{n}{2} \right\rfloor + 1$ leaving out one vertex unsaturated. Let $[V_1^i]$, $[V_2^i]$, $[V_3^i]$ and $[V_4^i]$ be the induced $P_3$-packing $4$-partition sets of $(CQ_6)_i$, $1 \le i \le 4$. One vertex $u_i$, $1 \le i \le 4$ in each $(CQ_6)_i$, $1 \le i \le 4$ is not included in any of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$ and $[V_4^i]$, $1 \le i \le 4$. For optimal induced $P_3$-packing $\left\lfloor \frac{n}{2} \right\rfloor + 1$ partition, it is necessary that the sub graph induced by $u_1$, $u_2$, $u_3$ and $u_4$ contains a path of length $2$ in $CQ_8$.
Consider $u_1$ in $(CQ_6)_1$, $deg(CQ_6)_1\,(u_1) = 6$. If $u_1$ is adjacent to vertices in $[V_1^1]$, $[V_2^1]$, $[V_3^1]$ and $[V_4^1]$, then the $4$-path containing $u_1$ cannot be included in any of $[V_1^1]$, $[V_2^1]$, $[V_3^1]$ and $[V_4^1]$, a contradiction. Suppose $u_1$ is adjacent to vertices in any one of $[V_i^1]$, $1 \le i \le 4$, then $ipp(CQ_6) > \left\lfloor \frac{n}{2} \right\rfloor + 1$ a contradiction. Hence $u_1$ is adjacent to $6$ vertices in at most two of $[V_i^1]$, $1 \le i \le 4$. This argument is also true for $u_i$ in $(CQ_6)_i$, $2 \le i \le 4$. This implies that the three vertices $u_1$, $u_2$ and $u_3$ are adjacent to at most four vertex sets. This implies $ipp(CQ_8) \ge \left\lfloor \frac{n}{2} \right\rfloor + 1$.
Assume the result is true for Crossed cube with dimension less than or equal to $n-1$. Consider $CQ_n$. When $n$ is even.
$CQ_n$ contains four copies of $CQ_{n-2}$, say $(CQ_{n-2})_1$, $(CQ_{n-2})_2$, $(CQ_{n-2})_3$ and $(CQ_{n-2})_4$. Let $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, …, $V^i_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, be the included $P_3$-packing $\left\lfloor \frac{n}{2} \right\rfloor + 1$ partition sets of $(CQ_{n-2})_i$, $1 \le i \le 4$. One vertex $u_i$, $1 \le i \le 4$ in each $(CQ_{n-2})_i$, $1 \le i \le 4$ is not included in any of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, …, $V^i_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, $1 \le i \le 4$.
For optimal induced $H$-packing $k$-partition, it is necessary that the sub graph induced by $u_1$, $u_2$, $u_3$ and $u_4$ contains a path of length $2$ in $CQ_n$. By the induction hypothesis, $ipp(CQ_{n-2})_1) \ge \left\lfloor \frac{n-2}{2} \right\rfloor + 1$ leaving out one vertex unsaturated. Label the vertices of $[V_1^i]$, $[V_2^i]$, $[V_3^i]$, …, $V^i_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$, in $(CQ_{n-2})_1$. Let φ be the mapping from $\{[V_1^1], [V_2^1], [V_3^1], …, V^1_{\left\lfloor \frac{n-4}{2} \right\rfloor}], V^1_{\left\lfloor \frac{n-2}{2} \right\rfloor}]\}$ to $\{1, 2, 3, …, \left\lfloor \frac{n-4}{2} \right\rfloor, \left\lfloor \frac{n-2}{2} \right\rfloor\}$ such that $\varphi([V_a]) = a$.
Similarly $ipp(CQ_{n-2})_2$ is greater than or equal $\left\lfloor \frac{n-2}{2} \right\rfloor + 1$, each of them leaving out one vertex unsaturated. Label the vertices of $[V_1^2]$, $[V_2^2]$, $[V_3^2]$, …, $V^2_{\left\lfloor \frac{n-2}{2} \right\rfloor}]$ in $(CQ_{n-2})_2$ as follows. Let φ be the mapping from $\{[V_1^2], [V_2^2], [V_3^2], …, V^2_{\left\lfloor \frac{n-4}{2} \right\rfloor}], V^2_{\left\lfloor \frac{n-2}{2} \right\rfloor}]\}$ to $\{1, 2, 3, …, \left\lfloor \frac{n-4}{2} \right\rfloor, \left\lfloor \frac{n-2}{2} \right\rfloor\}$ such that $\varphi([V_a]) = a + 1$. Let $u_1$ be the unsaturated vertex in $(CQ_{n-2})_1$. Then $|N(u_1)| = n-2$. By the induction hypothesis the edges incident at vertices of $N(u_1)$ are adjacent to vertices in at most $\left\lfloor \frac{n-4}{2} \right\rfloor$ partition sets. For the same reason $u_i$, $2 \le i \le 4$ is adjacent to vertices in at most $\left\lfloor \frac{n-2}{2} \right\rfloor$ partition sets. In $CQ_n$, the unsaturated vertex from each $(CQ_{n-2})_1$, $(CQ_{n-2})_2$, $(CQ_{n-2})_3$ and $(CQ_{n-2})_4$ induce a vertex disjoint path of length $2$, leaving out one vertex unsaturated. Hence $u_i$, $1 \le i \le 4$ is adjacent to at most $\left\lfloor \frac{n-2}{2} \right\rfloor + 1$ partition sets in $CQ_n$. Since $(CQ_{n-2})_1 \simeq (CQ_{n-2})_4$ and $(CQ_{n-2})_2 \simeq (CQ_{n-2})_3$, the three vertices $u_1$, $u_2$ and $u_3$ are adjacent to at most $\left\lfloor \frac{n-2}{2} \right\rfloor + 1$ partition sets. Therefore $ipp(CQ_n) \ge \left\lfloor \frac{n}{2} \right\rfloor + 1$. Suppose $n$ is odd. $CQ_n$ contains two copies of $CQ_{n-1}$, say $(CQ_{n-1})_1$ and $(CQ_{n-1})_2$. The induced $P_3$-packing $k$-partition number of $(CQ_{n-1})_1$ is $\left\lfloor \frac{n-2}{2} \right\rfloor + 1$ leaving out one vertex unsaturated. Since $(CQ_{n-1})_1$ is even. The role of the partition sets in

$(CQ_{n-1})_1$ is the same as that of $(CQ_{n-1})_2$. The union is an optimal induced $P_3$-packing $\left\lfloor \frac{n}{2} \right\rfloor + 1$partition leaving out two unsaturated vertices in $CQ_n$.

**Theorem 3.6**. $ipp(CQ_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, $n \geq 6$.

Proof. $[V_1^1]$, $[V_2^1]$, $[V_3^1]$,…, $[V_{\lfloor \frac{n-2}{2} \rfloor}^1]$, $[V_1^2]$, $[V_2^2]$, $[V_3^2]$, …, $[V_{\lfloor \frac{n-2}{2} \rfloor}^2]$, $[V_1^3]$, $[V_2^3]$, $[V_3^3]$,…, $[V_{\lfloor \frac{n-2}{2} \rfloor}^3]$ and $[V_1^4]$,$[V_2^4]$, $[V_3^4]$, …, $[V_{\lfloor \frac{n-2}{2} \rfloor}^4]$ be the partition sets of $(CQ_{n-2})_1$, $(CQ_{n-2})_2$, $(CQ_{n-2})_3$ and $(CQ_{n-2})_4$ leaving out one vertex unsaturated respectively. By previous lemma, the binding edges incident at vertices of $[V_i^1]$, $1 \leq i \leq \lfloor \frac{n-2}{2} \rfloor$, have their other ends in exactly in one $[V_j^2]$, $1 \leq j \leq \lfloor \frac{n-2}{2} \rfloor$in $CQ_{n-2}$. Without loss of generality we say that edges are between $[V_i^1]$ and $[V_{i+1}^1]$, $1 \leq i \leq \lfloor \frac{n-2}{2} \rfloor$ .The role of the partition sets in $(CQ_{n-2})_1$ is the same as that of $(CQ_{n-2})_4$ and the partition sets in $(CQ_{n-2})_2$ is the same as that of $(CQ_{n-2})_3$. By construction the four left out vertices induce a cycle $C$ in $CQ_n$. Let $P$ be a sub path of length $2$ of $C$ in $CQ_n$. The $\left\lfloor \frac{n-2}{2} \right\rfloor + 1$ partitions sets constructed by our method together with $P$ is an optimal induced $P_3$-packing $\left\lfloor \frac{n-2}{2} \right\rfloor + 1$ partition leaving out one vertex unsaturated in $CQ_n$.

**Theorem 3.7.** The Crossed Cube $CQ_n$, $n \geq 2$, Then $CQ_n$ has perfect $C_4$-packing.

Proof. Follows from Theorem 1.7.

**Lemma 3.8.** $ipp(CQ_3, C_4) = 2$.

Proof. Follows from Lemma 1.8.

**Lemma 3.9**. $ipp(CQ_n, C_4) = 4$.

Proof. Follows from Lemma 1.9.

## V. CONCLUSION

In this paper, we have proved that the induced $H$-packing $k$- partition problem where $H \simeq P_3$ exists for Enhanced Hypercube, Augmented cubes and Crossed cubes. Further we obtain $ipp(G, C_4)$ when $G$ is Enhanced hypercube, Augmented Cubes and Crossed Cubes networks . An induced $H$-packing $k$-partition for Generalized Exchanged Hypercubes, Folded Hypercubes, Twisted cubes, Spined cubes, Parity cubes and Petersen Cubes are under investigation.

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