Power Domination Parameters in Honeycomb-Like Networks

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Abstract A set S of vertices in a graph G is called a dominating set of G if every vertex in $V(G)\backslash S$ is adjacent to some vertex in S. A set S is said to be a power dominating set of G if every vertex in the system is monitored by the set S following a set of rules for power system monitoring. The power domination number of G is the minimum cardinality of a power dominating set of G. In this paper, we obtain the power domination number for triangular graphs, pyrene networks, circum-pyrene networks, circum-trizene networks, generalized honeycomb torus and honeycomb rectangular torus.

1 Introduction

Definition 1 ([1]) For $v \in V(G)$, the open neighbourhood of v, denoted as $N_G(v)$, is the set of vertices adjacent with v; and the closed neighbourhood of v, denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighbourhood of S is defined as $N_G(S) = \bigcup_{v \in S} N_G(v)$, and the closed neighbourhood of S is defined as $N_G[S] = N_G(S) \cup S$. For brevity, we denote $N_G(S)$ by $N(S)$ and $N_G[S]$ by $N[S]$.

Definition 2 ([1]) For a graph $G(V, E)$, $S \subseteq V$ is a dominating set of G if every vertex in $V\backslash S$ has at least one neighbour in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

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Definition 3 ([2]) Let $G(V, E)$ be a graph, and let $S \subseteq V(G)$. We define the sets $M^i(S)$ of vertices monitored by S at level i, $i \geq 0$, inductively as follows:

1. $M^0(S) = N[S]$.
2. $M^{i+1}(S) = M^i(S) \cup \{w : \exists v \in M^i(S), N(v) \cap (V(G) \backslash M^i(S)) = w\}$.

If $M^\infty(S) = V(G)$, then the set S is called a power dominating set of G. The minimum cardinality of a power dominating set in G is called the power domination number of G written $\gamma_p(G)$.

The power domination has been well studied for trees [1], product graphs [4], block graphs [3], interval graphs and so on. In fact, the problem has been shown to be NP-complete even when restricted to bipartite graphs and chordal graphs [1].

2 Main Results

In this section, we solve the power domination problem for triangular graphs, pyrene network, circum-pyrene network, circum-trizene network, generalized honeycomb torus and honeycomb rectangular torus. In 2013 Ferrero et al. [5] proved the following lemma which shows the power domination number for honeycomb mesh network $HM(n)$.

Lemma 1 *If G is the honeycomb mesh network $HM(n)$ of dimension n, then* $\gamma_p(G) \geq \left\lceil \frac{2n}{3} \right\rceil$.

The following lemma establishes a critical subgraph H of G in the sense that H contains at least one vertex of any power dominating set.

Lemma 2 *Let G be a graph and H as shown in Fig. 1a be a subgraph G with $deg_H w_i = deg_G w_i = 2$, $\forall i$, $i = 1, 2, 3, 4, 5, 6, 7, 8$. Then H is a critical subgraph of G.*

Proof Neither u nor v, when monitored, can further monitor any of w_i, $i = 1, 2, 3, 4, 5, 6, 7, 8$, as $deg_H u = deg_H v = 3$.

Definition 4 ([8]) Let n be a non-negative integer. A triangle graph of order n, TG_n, is defined in the following way: TG_1 is a hexagon. When $n \geq 2$, TG_n is built according to the following step:

Draw n rows of regular hexagons of the same size within an equilateral triangle (which is called the framework of TG_n) so that the first row consists of one hexagon, the second row consists of two hexagons and the nth row consists of n hexagons. Set all the vertices of these hexagons to be the vertices of TG_n, and set all the sides of these hexagons to be the edges of TG_n.

Lemma 3 *Let G be a triangle graph TG_n, $n \geq 2$. Then* $\gamma_p(G) \geq \left\lceil \frac{n}{2} \right\rceil$.

Proof In TG_n, there are $\left\lceil \frac{n}{2} \right\rceil$ critical subgraphs, each isomorphic to H as described in Lemma 2.2. Therefore, $\gamma_p(G) \geq \left\lceil \frac{n}{2} \right\rceil$.

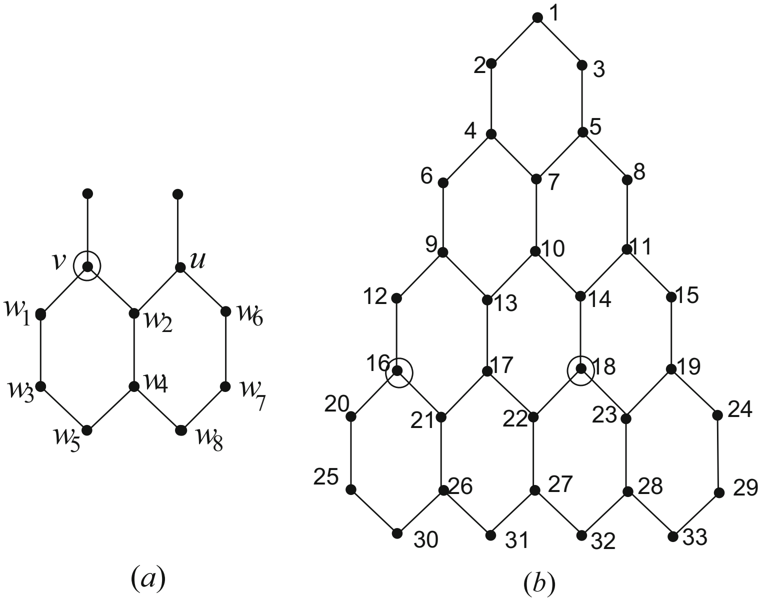

Fig. 1 (**a**) Circled vertices indicate a power dominating set of critical subgraph H induced by G (**b**) power dominating set of TG_4

Power Domination Algorithm in Triangular Graph

Input Triangular graph $TG_n,\ n \geq 2$.

Algorithm Name the vertices of $TG_n,\ n \geq 2$ as 1 to $n^2 + 4n + 1$ sequentially from left to right, row wise beginning with the top most row.

(i) Select $S_2 = \{4\}$ in TG_2.
(ii) Let $S_3 = \{9, 11\}$ in TG_3.
(iii) Inductively select $S_n = \bigcup_{k=1}^{\lceil \frac{n}{2} \rceil} n^2 + 2(k-1)$ in TG_n.

Output $\gamma_p(TG_n) = \lceil \frac{n}{2} \rceil$.

Proof of Correctness S_4 is a power dominating set of TG_4 with $|S_4| = 2$. Now $M^0(S_4) = N[S_4] = \{16, 20, 21, 12, 18, 22, 23, 14\}$. See Fig. 1b. At least one vertex $v \in M^0(S_4)$ satisfies $|N[v] \backslash M^0(S_4)| = 1$. Proceeding inductively, for every vertex $v \in M^i(S_4),\ |N[v] \backslash M^i(S_4)| = 1,\ i \geq 1$, at every inductive step $i, i \geq 1$. Now $S_n = \bigcup_{k=1}^{\lceil \frac{n}{2} \rceil} n^2 + 2(k-1)$ is a power dominating set of TG_n. This implies that $\gamma_p(TG_n) = \lceil \frac{n}{2} \rceil$, hence the proof.

Theorem 1 *Let G be a triangle graph TG_n. Then $\gamma_p(G) = \lceil \frac{n}{2} \rceil$.*

2.1 Power Domination in Pyrene Network

Pyrene is an alternante polycyclic aromatic hydrocarbon (PAH) and consists of four fused benzene rings, resulting in a large flat aromatic system. It is a colourless or pale yellow solid which forms during incomplete combustion of organic materials and therefore can be isolated from coal tar together with a broad range of related compounds. In the last four decades, a number of research works have been reported on both the theoretical and experimental investigation of pyrene concerning its electronic structure, UV-vis absorption and fluorescence emission spectrum. Indeed, this polycyclic aromatic hydrocarbon exhibits a set of many interesting electrochemical and photophysical attributes, which have resulted in its utilization in a variety of scientific areas. Like most PAHs, pyrene is used to make dyes, plastics and pesticides. Figure 2b depicts the graph of circum-pyrene (1). Circum-pyrene(2) is obtained by adding a layer of hexagons to the boundary of circum-pyrene (1). Inductively, circum-pyrene (n) is obtained from circum-pyrene($n-1$) by adding a layer of hexagons around the boundary of circum-pyrene ($n-1$). Similar construction follows for circum-trizene (n) [6]. See Fig. 3b.

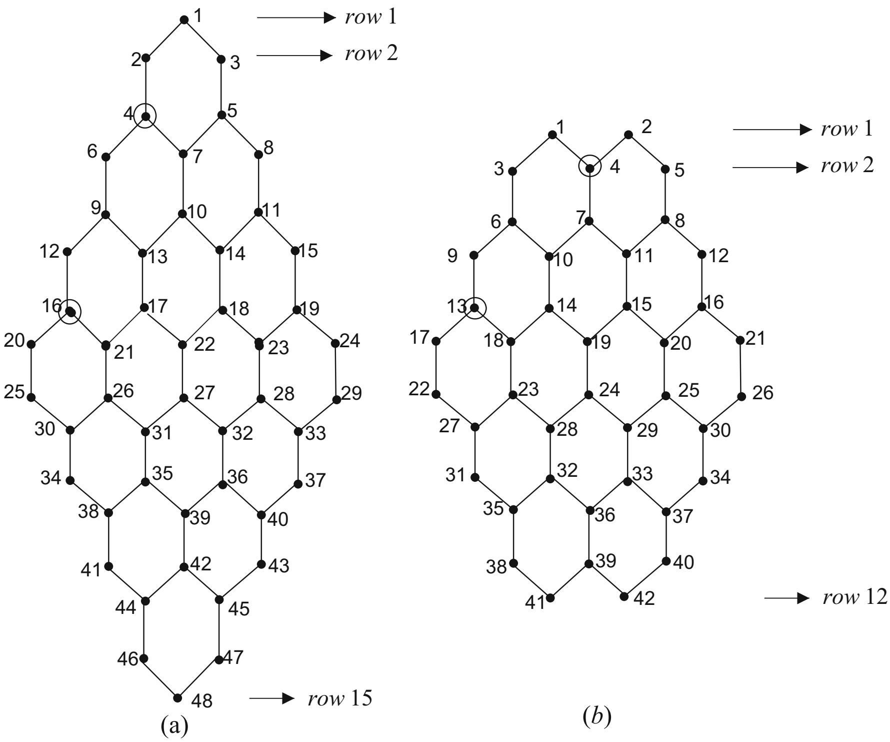

Fig. 2 (**a**) Power dominating set of $PY(4)$, (**b**) power dominating set of circum-pyrene(1)

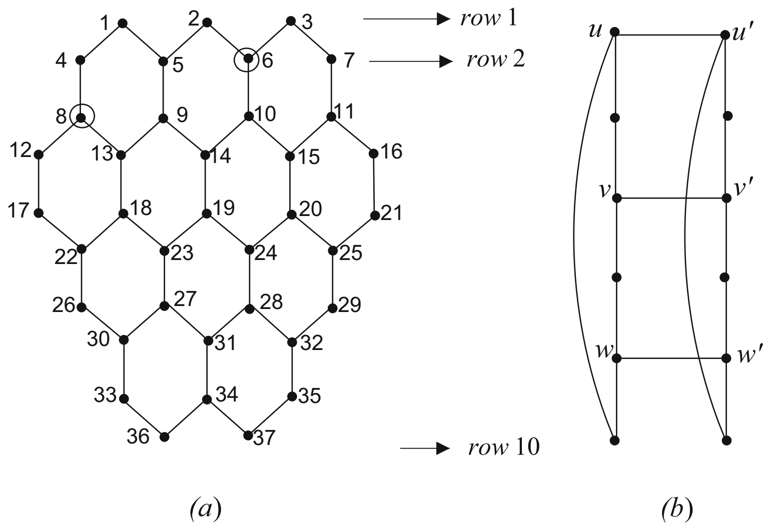

Fig. 3 (**a**) Circled vertices constitute a power dominating set of circum-trizene(1), (**b**) critical subgraph H of G

Lemma 4 *Let G be a pyrene network* $PY(n)$, $n \geq 4$. *Then* $\gamma_p(G) \geq \lceil \frac{n}{2} \rceil$.

Proof In $PY(n)$, there are $\lceil \frac{n}{2} \rceil$ critical subgraphs, each isomorphic to H as described in Lemma 2.2. Therefore, $\gamma_p(G) \geq \lceil \frac{n}{2} \rceil$.

Power Domination Algorithm in Pyrene Network

Input Pyrene network $PY(n)$, $n \geq 4$.

Algorithm Name the vertices of $PY(n)$, $n \geq 4$ as 1 to $2n^2 + 4n$ sequentially from left to right, row wise beginning with the topmost row. Let P^* denote the path induced by the edges of the hexagons that are not boundary edges of any other hexagon. Select $\lceil \frac{n}{2} \rceil$ vertices of degree 3 in P^*, which are at distance 4 apart on P^*. See Fig. 2a.

Output $\gamma_p(PY(n)) = \lceil \frac{n}{2} \rceil$.

Proof of Correctness S_4 is a power dominating set of $PY(4)$ with $|S_4| = 2$. Now $M^0(S_4) = N[S_4] = \{4, 6, 7, 2, 16, 20, 21, 12\}$. See Fig. 2a. At least one vertex $v \in M^0(S_4)$ satisfies $|N[v] \backslash M^0(S_4)| = 1$. Proceeding inductively, for every vertex $v \in M^i(S_4)$, $|N[v] \backslash M^i(S_4)| = 1$, $i \geq 3$, at every inductive step i, $i \geq 1$. Now $S_n = \bigcup_{i=1}^{\lceil \frac{n}{2} \rceil} (2i)^2$ is a power dominating set of $PY(n)$. This implies that $\gamma_p(PY(n)) = \lceil \frac{n}{2} \rceil$, hence the proof.

Theorem 2 *Let G be a pyrene network* $PY(n)$, $n \geq 4$. *Then* $\gamma_p(G) = \lceil \frac{n}{2} \rceil$.

Lemma 5 *Let G be a circum-pyrene*(n)*,* $n \geq 1$*. Then* $\gamma_p(G) \geq n+1$*.*

Proof In circum-pyrene(n), there are $2n+2$ critical subgraphs, each isomorphic to H as described in Lemma 2.2. Therefore, $\gamma_p(G) \geq \left\lceil \frac{2n+2}{2} \right\rceil = n+1$.

Power Domination Algorithm in Circum-Pyrene

Input Circum-pyrene(n), $n \geq 1$.

Algorithm Name the vertices of circum-pyrene(n), $n \geq 1$ as 1 to $6n^2 + 20n + 16$ sequentially from left to right, row wise beginning with the first row. Consider $2n+2$ hexagons in the outer most layer of the *circum-pyrene*(n). Let P^* denote the path induced by the edges of the hexagons that are not boundary edges of any other hexagon. Select $n+1$ vertices of degree 3 in P^*, which are at distance 5 apart on P^*.

Output $\gamma_p(\text{circum}-\text{pyrene}(n)) = n+1$.

Proof of Correctness $S(1)$ is a power dominating set of circum-pyrene(1) with $|S(1)| = 2$. Now $M^0(S(1)) = N[S(1)] = \{1, 2, 4, 7, 13, 17, 18, 9\}$. See Fig. 2b. At least one vertex $v \in M^0(S(1))$ satisfies $|N[v] \backslash M^0(S(1))| = 1$. Proceeding inductively, for every vertex $v \in M^i(S(1))$, $|N[v] \backslash M^i(S(1))| = 1$, $i \geq 1$, at every inductive step $i, i \geq 2$. Now $S(n) = n+1$ is a power dominating set of circum-pyrene(n). This implies that $\gamma_p(G) = n+1$, hence the proof.

Lemma 6 *Let G be a circum-trizene*(n)*,* $n \geq 1$*. Then* $\gamma_p(G) \geq n+1$*.*

Proof In circum-trizene(n), there are $2n+2$ critical subgraphs, each isomorphic to H as described in Lemma 2.2. Therefore, $\gamma_p(G) \geq \left\lceil \frac{2n+2}{2} \right\rceil = n+1$.

Power Domination Algorithm in Circum-Trizene

Input Circum-trizene(n), $n \geq 1$.

Algorithm Name the vertices of circum-trizene(n), $n \geq 1$ as 1 to $6n^2 + 18n + 13$ sequentially from left to right, row wise beginning with the first row. Consider $2n+2$ hexagons in the outer most layer of the *circum-trizene*(n). Let P^* denote the path induced by the edges of the hexagons that are not boundary edges of any other hexagon. Select $n+1$ vertices of degree 3 in P^*, which are at distance 5 apart on P^*.

Output $\gamma_p(\text{circum}-\text{trizene}(n)) = n+1$.

Proof of Correctness $S(1)$ is a power dominating set of circum-trizene(1) with $|S(1)| = 2$. Now $M^0(S(1)) = N[S(1)] = \{6, 2, 3, 10, 8, 4, 12, 13\}$. See Fig. 3a. At least one vertex $v \in M^0(S(1))$ satisfies $|N[v] \backslash M^0(S(1))| = 1$. Proceeding inductively, for every vertex $v \in M^i(S(1))$, $|N[v] \backslash M^i(S(1))| = 1$, $i \geq 1$, at every inductive step $i, i \geq 2$. Now $S(n) = n+1$ is a power dominating set of circum-trizene(n). This implies that $\gamma_p(G) = n+1$, hence the proof.

Theorem 3 *Let G be a circum-pyrene(n) or a circum-trizene(n), $n \geq 1$. Then* $\gamma_p(G) = n + 1$.

3 Ladderlike Honeycomb Networks

Lemma 7 *Let H be as shown in Fig. 3b. Then* $\gamma_p(H) = 1$.

Proof Let S be a power dominating set of H. We claim that $|S| = 1$. Suppose not, let H be the subgraph that does not contain any member of S. If any vertex of H is monitored, then v is adjacent to two unmonitored vertices of H, a contradiction. See Fig. 3b.

3.1 Honeycomb Rectangular Torus

Definition 5 ([7]) Assume that m and n are positive even integers. The honeycomb rectangular torus $HReT(m, n)$ is the graph with the node set $\{(i, j)\backslash 0 \leq i < m,\ 0 \leq j < n\}$ such that (i, j) and (k, l) are adjacent if they satisfy one of the following conditions:

1. $i = k$ and $j = l \pm 1(mod\ n)$; and
2. $j = l$ and $k = i - 1(mod\ m)$ if $i + j$ is even.

Definition 6 ([7]) Assume that m and n are positive integers where n is even. Let d be any integer such that $(m - d)$ is an even number. The generalized honeycomb rectangular torus $GHT(m, n, d)$ is the graph with the node set $\{(i, j)\backslash 0 \leq i < m,\ 0 \leq j < n\}$ such that (i, j) and (k, l) are adjacent if they satisfy one of the following conditions:

1. $i = k$ and $j = l \pm 1(modn)$
2. $j = l$ and $k = i - 1$ if $i + j$ is even and
3. $i = 0,\ k = m - 1$, and $l = j + d(modn)$ if j is even.

Obviously, any $GHT(m, n, d)$ is a three-regular bipartite graph. We can label those nodes (i, j) white when $i + j$ is even or black otherwise.

Lemma 8 *Let G be a* $HReT(m, n)$, m, n *are even* $m \geq 6,\ n \geq 8$ *and* $m \leq n$. *Then* $\gamma_p(G) \geq \frac{n}{2}$.

Proof In $HReT(m, n)$, there are $\frac{n}{2}$ critical subgraphs, each isomorphic to H as described in Lemma 3.1. Therefore, $\gamma_p(G) \geq \frac{n}{2}$.

Power Domination Algorithm in Honeycomb Rectangular Torus

Input The honeycomb rectangular torus $HReT(m, n)$, m, n is even $m \geq 6,\ n \geq 8$ and $m \leq n$.

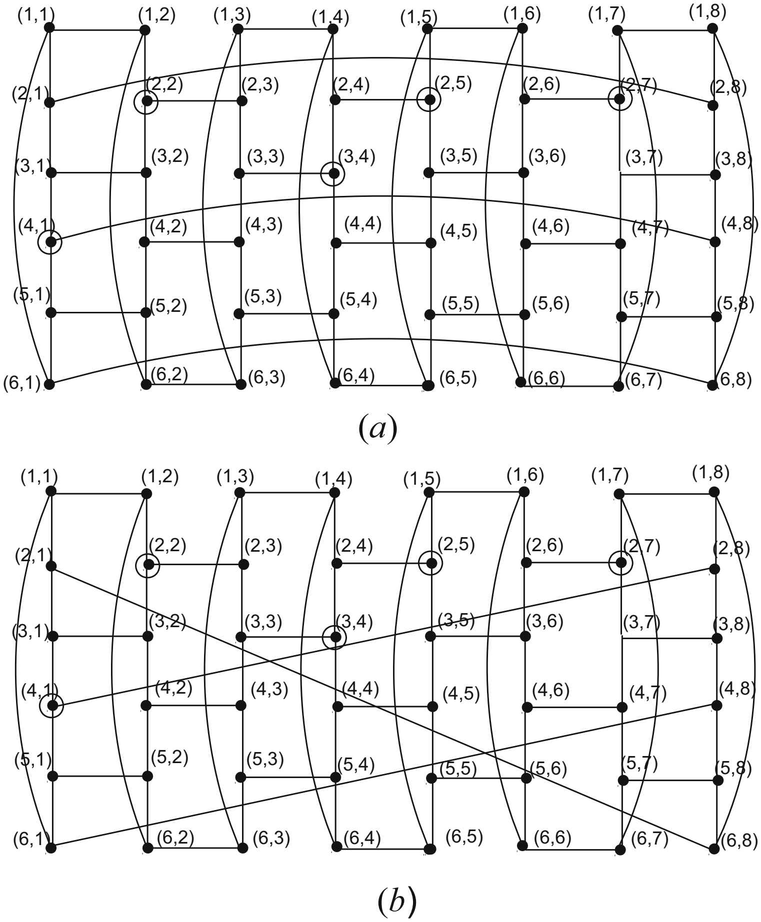

Fig. 4 Circled vertices constitute a power dominating set. (**a**) Honeycomb rectangular torus $HReT(6, 8)$. (**b**) Honeycomb rectangular torus $GHT(6, 8, 2)$

Algorithm Name the vertices in the ith row, jth column position as (i, j), $1 \leq i \leq m$, $1 \leq j \leq n$, and select the vertices $\bigcup_{j=5}^{n-1}(2, j) \cup \{(2, 2), (3, 4), (4, 1)\}$ in S.

Output $\gamma_p(HReT(m, n)) = \frac{n}{2} + 1$.

Proof of Correctness Let S be a power dominating set of $HReT(m, n)$ with $|S| = \frac{n}{2} + 1$. Then $M^0(S) = N[v] = \{(i, j), (2, k), (2, 2), (1, 2), (3, 2), (2, 3), (3, 4), (4, 4), (4, 1), (5, 1), (3, 1), (4, n)\}$, $i = 1, 2, 3$, $j = 5, 7, \ldots, n-1$, $k = 4, 6, \ldots, n-2$. See Fig. 4a. At least one vertex $v \in M^0(S(1))$ satisfies $|N[v] \backslash M^0(S(1))| = 1$. Proceeding inductively, for every vertex $v \in M^i(S)$, $|N[v] \backslash M^i(S)| = 1$, $i \geq 1$, at every inductive step $i, i \geq 1$.

Now $S = \bigcup_{j=5}^{n-1}(2, j) \cup \{(2, 2), (3, 4), (4, 1)\}$ is a power dominating set of $(HReT(m, n))$. This implies that $\gamma_p(G) = \frac{n}{2} + 1$, hence the proof.

Lemma 9 *Let G be a generalized honeycomb rectangular torus $GHT(m, n, d)$, $m \geq 6,\ n \geq 8,\ m \leq n$. Then $\gamma_p(G) \geq \frac{n}{2}$.*

Proof In $GHT(m, n, d)$, there are $\frac{n}{2}$ vertex-disjoint copies of H as described in lemma 3.1. Therefore, $\gamma_p(G) \geq \frac{n}{2}$.

Theorem 4 *Let G be a honeycomb rectangular torus $HReT(m, n)$ or a generalized honeycomb rectangular torus $GHT(m, n, d)$, $m \geq 6,\ n \geq 8, m \leq n$. Then $\frac{n}{2} \leq \gamma_p(G) \leq \frac{n}{2} + 1$.*

4 Conclusion

In this paper, we have obtained the power domination number for triangular graphs, pyrene networks, circum-pyrene networks, circum-trizene networks, honeycomb rectangular torus and generalized honeycomb torus network.

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