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Generalized Bernoulli Numbers and Polynomials in the Context of the Clifford Analysis

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In this paper, we consider the generalization of the Bernoulli numbers and polynomials for the case of the hypercomplex variables. Multidimensional analogs of the main properties of classic polynomials are proved.

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Introduction

The Bernoulli polynomials for natural values of the argument were first considered by J. Bernoulli (1713) in relation to the problem of summation of powers of consecutive natural numbers. L. Euler studied such polynomials for arbitrary values of the argument, the term "Bernoulli polynomials" was introduced by J. L. Raabe (1851).

The Bernoulli numbers and polynomials are well studied and find applications in fields of pure and applied mathematics. Various variants of generalization of the Bernoulli numbers and polynomials can be found in [5–11]. A generalization to several variables has been considered in [12]; in this paper definitions of the Bernoulli numbers and polynomials associated with rational lattice cones were given and multidimensional analogs of their basic properties were proved.

This paper is devoted to generalization of these results to the case of hypercomplex variables. The Clifford algebra in hypercomplex function theory (HFT) was first used by R. Fueter [1] in the beginning of the last century. A systematic study of this topic can be found in [2–4]. Also, the papers [15–18] with further advancement of the Clifford analysis should be noted. The notion of the Bernoulli numbers and polynomials in this framework were given and studied in [13, 14]. In this paper we give a more genral notion of Bernoulli polynomials than in [13, 14], namely, in the spirit of [12] we define polynomials in hypercomplex variables associated with a matrix of integers. In the second section of the paper we formulate and prove basic properties of such polynomials.

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1. Notation and definitions of the Generalized Bernoulli Polynomials and Bernoulli Numbers

Let $\{e_1, \ldots, e_n\}$ be an orthonormal base of the Euclidean vector space $\mathbb{R}^n$ with a product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl}, \quad k, l = 1, \ldots, n,$$

where δ_{kl} is the Kronecker symbol. This non commutative product generates the 2^n-dimensional Clifford algebra $Cl_{0,n}$ over $\mathbb{R}$ and the set$(e_A : A \subseteq \{1, \ldots, n\})$ with $e_A = e_{h_1} e_{h_2} \ldots e_{h_r},\ 1 \leqslant h_1 \leqslant \ldots \leqslant h_n,\ e_\phi = e_0 = 1$, forms a basis of $Cl_{0,n.}$ The real vector space $\mathbb{R}^{n+1}$ is embedded in $Cl_{0,n}$ by identifying the element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with

$$z = x_0 e_0 + x_1 e_1 + \cdots + x_n e_n \in A \equiv \text{span}_{\mathbb{R}}\ \{e_0, \ldots, e_n\} \cong\ \mathbb{R}^{n+1}.$$

The natural generalization of the complex Cauchy-Riemann operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

is given by the operator

$$D = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + \ldots + \frac{\partial}{\partial x_n} e_n,$$

and the equation

$$Df = 0$$

defines hypercomplex holomorphic (or monogenic) functions $f = f(z)$ as Clifford algebra valued functions in the kernel of this generalized Cauchy-Riemann operator (cf. [15]). Since the operator D can be applied both from the left and from the right hand side of f, it is usual to refer to it as a left monogenic function or a right monogenic function, respectively. For simplicity, from now on we only deal with left monogenic functions. The case of right monogenic functions can be treated completely analogously.

Since $Dz = 1 - n$ it is evident that the function $f(z) = z \in \mathcal{A}$ is only monogenic if $n = 1$, i.e., in the case of $\mathcal{A} = \mathbb{C}$. This implies significant differences between the cases $n = 1$ and $n > 1$. Moreover, powers of z, i.e., $f(z) = z^k,\ k = 2, \ldots,$ are not monogenic which means that they cannot be considered appropriate as hypercomplex generalizations of the complex power $z^k,\ z \in \mathbb{C}$. These facts are the reason for generalized power series of a special structure, which we are going to use in the following subsection.

To overcome the mentioned situation, in [16] has been considered another hypercomplex structure for $\mathbb{R}^{n+1}$ and

$$\mathcal{H}^n = \{\vec{z} : \vec{z} = (z_1, \ldots, z_n),\ z_k = x_0 - x_k e_k,\ x_0,\ x_k \in \mathbb{R}\ k = 1, \ldots, n\},$$

whereas the components of the vector $\vec{z}$, i.e. the hypercomplex variables z_k themselves are monogenic, their ordinary products $z_i z_k,\ i \neq k$, are not monogenic. But a n-ary operation, namely their permutational (symmetric) product resolves the problem (cf. [16]).

Definition 1. *Let V_+ be a commutative or non-commutative ring, $a_k \in V\ (k = 1, \ldots, n)$, then the symmetric "$\times$" product is defined by*

$$a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\pi(i_1, \ldots, i_n)} a_{i_1} a_{i_2} \cdots a_{i_n}, \tag{1}$$

*where the sum runs over **all** permutations of all $(i_1, \ldots, i_n)$.*

Additionally, the following convention has been introduced in [16].

Convention. *If the factor a_j occurs σ_j times in (1), we briefly write*

$$\underbrace{a_1 \times \cdots \times a_1}_{\sigma_1} \times \cdots \times \underbrace{a_n \times \cdots \times a_n}_{\sigma_n} = a_1^{\sigma_1} \times \cdots \times a_n^{\sigma_n} = \vec{a}^{\sigma}, \tag{2}$$

where $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}_0^n$ and set parentheses if the powers are understood in the ordinary way.

Formula (2) simply allows to work with a polynomial formula exactly in the same way as in the case of several commutative variables. It holds

$$(z_1 + \cdots + z_n)^k = \sum_{|\sigma|=k} \begin{pmatrix} k \\ \sigma \end{pmatrix} z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n} = \sum_{|\sigma|=k} \begin{pmatrix} k \\ \sigma \end{pmatrix} \vec{z}^{\sigma}, \ k \in \mathbb{N} \tag{3}$$

with polynomial coefficients defined as usual by $\begin{pmatrix} k \\ \sigma \end{pmatrix} = \dfrac{k!}{\sigma!(k-\sigma)!}$, where $\sigma! = \sigma_1! + \cdots + \sigma_n!$ (see [17, 18]).

In [17] it has been shown that the partial derivatives of $\vec{z}^{\sigma}$ with respect to x_k are obtained as

$$\frac{\partial \vec{z}^{\sigma}}{\partial x_k} = \sigma_k \vec{z}^{\sigma - \tau_k}, \tag{4}$$

where τ_k is the multiindex with 1 at place k and zero otherwise.

It is well known that for complex holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ the complex derivative $f' = \dfrac{df}{dz}$ exists and coincides with the complex partial derivative

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right).$$

The analogous situation is true in the hypercomplex case (cf. [18]). A real differentiable function $f(\vec{z})$ is left (right) hypercomplex derivable in $\Omega \subset \mathcal{H}^n$ if and only if f is left (right)monogenic in $\Omega \subset \mathcal{H}^n$. In the case of its existence, the hypercomplex derivative is given by

$$\frac{1}{2}\overline{D}f \ \ resp. \frac{1}{2}f\overline{D},$$

with the conjugated generalized Cauchy-Riemann operator

$$\overline{D} = \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}e_1 - \cdots - \frac{\partial}{\partial x_n}e_n.$$

Furthermore, like in the complex case, where the complex derivative satisfies

$$f' = \frac{df}{dz} = \frac{\partial f}{\partial x},$$

the left(right) hypercomplex derivative of f at $\vec{z}$ is exactly

$$\frac{1}{2}\overline{D}f = \frac{1}{2}f\overline{D} = \frac{\partial f}{\partial x_0}.$$

Let $a^1, \ldots, a^n$ be vectors with real coordinates $a^j = (a_1^j, \ldots, a_n^j)$ and

$$A = \begin{pmatrix} a_1^1 & \ldots & \ldots & a_1^n \\ a_2^1 & \ldots & \ldots & a_2^n \\ \vdots & \vdots & \vdots & \\ a_n^1 & \ldots & \ldots & a_n^n \end{pmatrix}$$

is a matrix with coordinates of the vectors a^j in a column.

Definition 2. *The hypercomplex Bernoulli polynomials* $B_\mu(z) = B_{\mu_1}, \dots, \mu_n(z_1, \dots, z_n)$, $\mu_k \in \mathbb{N}_0,\ k = 1, \dots, n$ *associated with the matrix* A *are defined as the coefficients of a multiple power series ordered with respect to the degree of homogenity by the following relation*

$$e^{<\vec{t},\vec{z}>} = \prod_{j=1}^{n} \left(\sum_{k=0}^{\infty} \frac{< a^j, t >^k}{(k+1)!} \right) \sum_{|\mu|=0}^{\infty} B_\mu(z)\ \frac{t^\mu}{\mu!}, \tag{5}$$

where $< t, z > =\ t_1 z_1 + \ldots + t_n z_n,\quad z = (z_1, \dots, z_n),\ t^\mu = t_1^{\mu_1} \dots t_n^{\mu_n},\ \mu! = \mu_1! \dots \mu_n!,$
$| \mu |= \mu_1 + \ldots, +\mu_n$.

Applying (3), the formula (5) is equivalent to

$$\sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!}\ z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n}\ t^\sigma = \sum_{|s|=0}^{\infty} \frac{1}{(|\ s\ | + 1)s!} (a^j t)^s \ \sum_{|\mu|=0}^{\infty} B_\mu(z)\ \frac{t^\mu}{\mu!},$$

where $s = (s_1, \dots, s_n)$, $\sigma = (\sigma_1, \dots, \sigma_n)$ and $(a^j t)^s = \left(a_1^j t_1\right)^{s_1} \dots \ \left(a_n^j t_n\right)^{s_n}$.

Comparing both sides gives the relationship of hypercomplex Bernoulli polynomials to the generalized powers

$$\sum_{\alpha+\mu=\sigma} \ \sum_{s^1+\ldots+s^n=\alpha} \left(\prod_{j=1}^{n} \frac{1}{(|\ s^j\ |+1)s^j!} \right) (a^1 t)^{s^1} \dots \ (a^n t)^{s^n} \frac{B_\mu(z)}{\mu!} = \frac{1}{\sigma!}\ {z_1}^{\sigma_1} \times {z_2}^{\sigma_2} \times \cdots \times {z_n}^{\sigma_n}, \tag{6}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\mu = (\mu_1, \dots, \mu_n)$, $t = (t_1, \dots, t_n)$, $z = (z_1, \dots, z_n)$, $\sigma = (\sigma_1, \dots, \sigma_n)$, $s^j = (s_1^j, \dots, s_n^j)$ for $\sigma_k = 0, 1, \dots\ (k = 1, \dots, n)$.

Obviously, the set of hypercomplex Bernoulli polynomials contains n copies of the classical Bernoulli polynomials that are obtained when matrix $A = \begin{pmatrix} 1 & \dots & 1 \\ \dots & \dots & \dots \\ 1 & \dots & 1 \end{pmatrix}$ and all the indices $\mu_k,\ k = 1, \dots, n$ in (6) are equal to zero or only one of them is different from zero.

For example, some hypercomplex Bernoulli polynomials given by (6), for $n = 2$, are equal to

$$B_{0,0}(z) = 1,$$
$$B_{1,0}(z) = z_1 - \frac{1}{2}\left(a_1^1 + a_1^2\right),$$
$$B_{0,1}(z) = z_2 - \frac{1}{2}\left(a_2^1 + a_2^2\right),$$
$$B_{1,1}(z) = z_1 \times z_2 - \frac{1}{2}\Big[z_1\left(a_2^1 + a_2^2\right) + z_2\left(a_1^1 + a_1^2\right)\Big] + \frac{1}{4}\left(a_1^1 a_2^2 + a_2^1 a_1^2\right) + \frac{1}{6}\left(a_1^1 a_2^1 + a_1^2 a_2^2\right),$$
$$B_{2,0}(z) = z_1^2 - z_1\left(a_1^2 + a_1^1\right) + \frac{1}{6}\Big((a_1^1)^2 + (a_1^2)^2\Big) + \frac{1}{2} a_1^1 a_1^2.$$

Definition 3. *Generalized Bernoulli numbers* $B_{\sigma_1,\dots,\sigma_n}$ *are the values of the Bernoulli polynomials at the origin*

$$B_{\sigma_1,\dots,\sigma_n} = B_{\sigma_1,\dots,\sigma_n}(0, \dots, 0).$$

For instance, for $n = 2$ we obtain next values of Bernoulli numbers:

$$\begin{aligned} B_{0,0} =& B_{0,0}(0,0) = 1, \\ B_{1,0} =& B_{1,0}(0,0) = -\frac{1}{2}\big(a_1^1 + a_1^2\big), \\ B_{0,1} =& B_{0,1}(0,0) = -\frac{1}{2}\big(a_2^1 + a_2^2\big), \\ B_{1,1} =& B_{1,1}(0,0) = \frac{1}{4}\big(a_1^1 a_2^2 + a_2^1 a_1^2\big) + \frac{1}{6}\big(a_1^1 a_2^1 + a_1^2 a_2^2\big), \\ B_{2,0} =& B_{2,0}(0,0) = \frac{1}{6}\Big((a_1^1)^2 + (a_1^2)^2\Big) + \frac{1}{2} a_1^1 a_1^2. \end{aligned}$$

2. Properties of Bernoulli numbers and Bernoulli polynomials

Property 1. *Hypercomplex Bernoulli polynomials and generalized Bernoulli numbers satisfy the expression*

$$B_{\sigma_1,\dots,\sigma_n}(1,\dots,1) = (-1)^{|\sigma|} B_{\sigma_1,\dots,\sigma_n}. \tag{7}$$

Proof. Making use of the definition of hypercomplex Bernoulli polynomials by generating function

$$F(\vec{t},\vec{z}) = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n}(z_1,\dots,z_n)\ t_1^{\sigma_1}\dots t_n^{\sigma_n},$$

where

$$F(\vec{t},\vec{z}) = \prod_{j=1}^{n} \frac{< a^j, t >}{e^{< a^j,t >} - 1} e^{< z,t >},$$

and taking $(z_1,\dots,z_n) = (0,\dots,0)$ and $(z_1,\dots,z_n) = (1,\dots,1)$ we get

$$F(\vec{t},\vec{0}) = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n} t_1^{\sigma_1}\cdot\dots\cdot t_n^{\sigma_n}$$

and

$$F(\vec{t},\vec{1}) = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n}(1,\dots,1)\ t_1^{\sigma_1}\cdot\dots\cdot t_n^{\sigma_n},$$

respectively.

Moreover $F(\vec{t},\vec{1}) = F(-\vec{t},\vec{0})$, that is,

$$\begin{aligned} \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n}(1,\dots,1)\ t_1^{\sigma_1}\dots t_n^{\sigma_n} &= \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n}\ (-t_1)^{\sigma_1}\dots(-t_n)^{\sigma_n} = \\ &= \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n}\ (-1)^{|\sigma|}\ t_1^{\sigma_1}\dots t_n^{\sigma_n}. \end{aligned}$$

Hence,

$$B_{\sigma_1,\dots,\sigma_n}(1,\dots,1) = (-1)^{|\sigma|}\ B_{\sigma_1,\dots,\sigma_n}.$$

□

The equality (7) generalizes the property $B_n(1) = (-1)^n B_n,\ n \in \mathbb{N}_0$, already known in the classical case.

Property 2. *Hypercomplex Bernoulli polynomials can be expressed by generalized Bernoulli numbers as*

$$B_{\sigma_1,\dots,\sigma_n}(z_1,\dots,z_n) = \sum_{j_1=0}^{\sigma_1}\cdots\sum_{j_n=0}^{\sigma_n}\binom{\sigma_1}{j_1}\cdots\binom{\sigma_n}{j_n}B_{j_1,\dots,j_n}z_1^{\sigma_1-j_1}\times\cdots\times z_n^{\sigma_n-j_n}. \quad (8)$$

Proof. Using the definitions of hypercomplex Bernoulli polynomials and the Bernoulli numbers, we can write

$$\sum_{|\sigma|=0}^{\infty}\left(\sum_{j+k=\sigma}B_{j_1,\dots,j_n}\frac{z_1^{k_1}\times\cdots\times z_n^{k_n}}{j_1!\cdots j_n!k_1!\cdots k_n!}\right)t_1^{\sigma_1}\cdots t_n^{\sigma_n} = \sum_{|\sigma|=0}^{\infty}\frac{B_{\sigma_1,\dots,\sigma_n}(z_1,\dots,z_n)}{\sigma_1!\dots\sigma_n!}t_1^{\sigma_1}\cdots t_n^{\sigma_n},$$

which yields

$$B_{\sigma_1,\dots,\sigma_n}(z_1,\dots,z_n) = \sum_{j_1=0}^{\sigma_1}\cdots\sum_{j_n=0}^{\sigma_n}B_{j_1,\dots,j_n}\frac{\sigma_1!\dots\sigma_n!z_1^{\sigma_1-j_1}\times\cdots\times z_n^{\sigma_n-j_n}}{j_1!\dots j_n!(\sigma_1-j_1)!\dots(\sigma_n-j_n)!},$$

that is,

$$B_{\sigma_1,\dots,\sigma_n}(z_1,\dots,z_n) = \sum_{j_1=0}^{\sigma_1}\cdots\sum_{j_n=0}^{\sigma_n}\binom{\sigma_1}{j_1}\cdots\binom{\sigma_n}{j_n}B_{j_1,\dots,j_n}z_1^{\sigma_1-j_1}\times\cdots\times z_n^{\sigma_n-j_n}. \qquad \square$$

With (8) we found a generalization for another property of the classical Bernoulli polynomials $B_n(z)$:

$$B_n(z) = \sum_{k=0}^{n}\binom{n}{k}B_k z^{n-k},\ n\in\mathbb{N}_0.$$

Proposition 2 still allows to introduce a new type of Bernoulli numbers, where one of the arguments is equal to one and the others are equal to zero, which is a situation different from that one in Proposition 1, which describes the symmetry relation between $B_{\sigma_1,\dots,\sigma_n}(1,\dots,1)$ and $B_{\sigma_1,\dots,\sigma_n}$.

Property 3. *Let us call k-Bernoulli numbers, $B^k_{\sigma_1,\dots,\sigma_n}$, those that are obtained by calculating the hypercomplex Bernoulli polynomials in $(0,\dots,\underbrace{1}_{k},\dots,0)$, $k=1,\dots,n$, i.e.,*

$$B^k_{\sigma_1,\dots,\sigma_n} = B_{\sigma_1,\dots,\sigma_n}(0,\dots,\underbrace{1}_{k},\dots,0).$$

Then these k-Bernoulli numbers can be represented as linear combinations of the generalized Bernoulli numbers,

$$B^k_{\sigma_1,\dots,\sigma_n} = \sum_{j_k=0}^{\sigma_k}\binom{\sigma_k}{j_k}B_{\sigma_1,\dots,j_k,\dots,\sigma_n}.$$

Proof. The proof follows immediately from (8) by taking $z_k=1$ and $z_i=0$, $i=1,\dots,n$, $i\neq k$. $\square$

Example.

$$B^1_{2,1}\equiv B_{2,1}(1,0) = \mathbf{1}B_{0,1}+\mathbf{2}B_{1,1}+\mathbf{1}B_{2,1}$$

$$B^1_{3,2}\equiv B_{3,2}(1,0) = \mathbf{1}B_{0,2}+\mathbf{3}B_{1,2}+\mathbf{3}B_{2,2}+\mathbf{1}B_{3,2}$$

$$B^1_{4,3}\equiv B_{4,3}(1,0) = \mathbf{1}B_{0,3}+\mathbf{4}B_{1,3}+\mathbf{6}B_{2,3}+\mathbf{4}B_{3,3}+\mathbf{1}B_{4,3}$$

$$\ldots \quad \ldots$$

$$B_{1,1}^2 \equiv B_{1,1}(0,1) = \mathbf{1}B_{1,0} + \mathbf{1}B_{1,1}$$

$$B_{1,2}^2 \equiv B_{1,2}(0,1) = \mathbf{1}B_{1,0} + \mathbf{2}B_{1,1} + \mathbf{1}B_{1,2}$$

$$B_{2,3}^2 \equiv B_{2,3}(0,1) = \mathbf{1}B_{2,0} + \mathbf{3}B_{2,1} + \mathbf{3}B_{2,2} + \mathbf{1}B_{2,3}$$

Property 4. *We have*

$$\frac{\partial}{\partial x_k} B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \begin{cases} \sigma_k B_{\sigma_1,\ldots,\sigma_k-1,\ldots,\sigma_n}(z_1,\ldots,z_n) & ,\sigma_k \neq 0 \\ 0 & ,\sigma_k = 0, \end{cases} \quad k = 1,\ldots,n.$$

where $z_k = x_k - x_0 e_k,\ x_0, x_k \in \mathbb{R},\ k = 1,\ldots,n$

Proof. The proof follows directly by partial differentiation with respect to x_k of both sides of (8) together with (4). □

This proposition generalizes for the hypercomplex case the relations $B_n'(z) = nB_{n-1}(z)$, $n \in \mathbb{N}$, used for the differentiation of classical Bernoulli polynomials.

Property 5. *For the hypercomplex derivative of* $B_\sigma(z)$ *of* $\frac{1}{2}\overline{D}B_\sigma(z)$ *the next property holds true*

$$\frac{1}{2}\overline{D}B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = -\sum_{k=1}^{n} \sigma_k B_{\sigma_1,\ldots,\sigma_k-1,\ldots,\sigma_n}(z_1,\ldots,z_n)e_k.$$

Proof. Considering that the hypercomplex Bernoulli polynomials are monogenic, i.e.,

$$DB_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = 0,$$

we can write

$$\frac{\partial}{\partial x_0} B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k} B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)e_k,$$

that is

$$\frac{1}{2}\overline{D}B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = -\sum_{k=1}^{n} \sigma_k B_{\sigma_1,\ldots,\sigma_k-1,\ldots,\sigma_n}(z_1,\ldots,z_n)e_k.$$

□

Let $\delta = (\delta_1,\ldots,\delta_n),\ \delta_j$ is a shift operator $\delta_j f(z) = f(z_1,\ldots,z_j+1,\ldots,z_n)$.

To formulate the next property we have to introduce some notation. Denote by $Q(\delta) = \prod_{j=1}^{n}(\delta^{a^j} - 1)$, the linear operator where $\delta^{a^j} = \delta_1^{a_1^j}\ldots,\delta_n^{a_n^j}$. Denote by $\partial_{a_j} = <a^j, \partial> = \sum_{k=1}^{n} a_k^j \partial_k$ the differential operator along the vector a^j, and let $\partial_a = \partial_{a_1}\ldots\partial_{a_n}$.

Property 6. *Hypercomplex Bernoulli polynomials satisfy the equation*

$$Q(\delta)B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \partial_a z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n}.$$

Proof. Note that $(\delta^{a^j} - 1)e^{<\vec{z},\vec{t}>} = (e^{<a^j,t>}, -1)e^{<\vec{z},\vec{t}>}$, then

$$\prod_{j=1}^{n}(\delta^{a^j} - 1)e^{<\vec{z},\vec{t}>} = \prod_{j=1}^{n}(e^{<a^j,t>} - 1)e^{<\vec{z},\vec{t}>}.$$

It means that operator $Q(\delta)$ acts on $e^{<\vec{z},\vec{t}>}$ by the formula

$$Q(\delta)e^{<\vec{z},\vec{t}>} = \prod_{j=1}^{n}(e^{<a^j,t>} - 1)e^{<\vec{z},\vec{t}>}. \tag{9}$$

Now we will show how the operator $Q(\delta)$ acts on the generating function $F(\vec{t},\vec{z})$ of Bernoulli polynomials.

$$Q(\delta)F(\vec{t},\vec{z}) = \prod_{j=1}^{n}\frac{< a^j, t >}{e^{<a^j,t>} - 1}Q(\delta)e^{<\vec{z},\vec{t}>}.$$

Using (9) we obtain

$$Q(\delta)F(\vec{t},\vec{z}) = \prod_{j=1}^{n}\frac{< a^j, t >}{e^{<a^j,t>} - 1}\ \prod_{j=1}^{n}(e^{<a^j,t>} - 1)e^{<\vec{z},\vec{t}>}.$$

As a result we have

$$Q(\delta)F(\vec{t},\vec{z}) = \prod_{j=1}^{n} < a^j, t > e^{<\vec{z},\vec{t}>}.$$

Since $\partial_{a_j}e^{<\vec{z},\vec{t}>} = < a^j, t > e^{<\vec{z},\vec{t}>}$, then

$$\partial_a e^{<\vec{z},\vec{t}>} = \prod_{j=1}^{n} < a^j, t > e^{<\vec{z},\vec{t}>}.$$

Using the definition of Bernoulli polynomials and acting by operator $Q(\delta)$ on each part, we obtain

$$Q(\delta)\prod_{j=1}^{n}\frac{< a^j, t >}{e^{<a^j,t>} - 1}e^{<\vec{z},\vec{t}>} = Q(\delta)\sum_{|\sigma|=0}^{\infty}\frac{B_{\sigma_1,\dots,\sigma_n}(z_1,\dots,z_n)\ t^{\sigma}}{\sigma_1!\dots\sigma_n!}.$$

Since $Q(\delta)F(\vec{t},\vec{z}) = \partial_a e^{<\vec{z},\vec{t}>}$, then

$$\partial_a e^{<\vec{z},\vec{t}>} = \sum_{|\sigma|=0}^{\infty}\frac{Q(\delta)\ B_{\sigma_1,\dots,\sigma_n}(z_1,\dots,z_n)\ t^{\sigma}}{\sigma_1!\dots\sigma_n!}.$$

$$\sum_{|\sigma|=0}^{\infty}\frac{\partial_a\ z_1^{\sigma_1}\times\dots\times z_n^{\sigma_n}}{\sigma_1!\dots\sigma_n!}t^{\sigma} = \sum_{|\sigma|=0}^{\infty}\frac{Q(\delta)\ B_{\sigma_1,\dots,\sigma_n}(z_1,\dots,z_n)\ t^{\sigma}}{\sigma_1!\dots\sigma_n!}.$$

$$Q(\delta)B_{\sigma_1,\dots,\sigma_n}(z_1,\dots,z_n) = \partial_a z_1^{\sigma_1}\times\dots\times z_n^{\sigma_n}.$$

□

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Обобщенные числа и полиномы Бернулли в контексте Клиффордова анализа

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В работе рассматривается обобщение чисел и многочленов Бернулли для случая гиперкомплексных переменных. Доказаны многомерные аналоги основных свойств классических чисел и многочленов Бернулли.

Ключевые слова: гиперкомплексные многочлены Бернулли, производящие функции, клиффордов анализ.