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Laplacian and Effective Resistance Metric in Sierpinski Gasket Fractal

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Abstract Laplacian operator for functions on fractal field plays a vital role in the study of partial differential equations of nonlinear in fractals. In this paper self-similar fractal Sierpinski gasket is considered with regular harmonic structures, and energy renormalization factor and scaling constant are obtained. Effective resistance presents a metric with which the properties of the fractal and the transmission can be discussed. Hausdorff dimension of Sierpinski gasket fractal is obtained by scaling constant. Spectral dimension of Sierpinski gasket fractal is calculated by using Laplacian and effective resistance metric. Finally the dimensions of the Sierpinski gasket are interpreted.

Keywords Energy renormalization factor · Sierpinski gasket fractal · Effective resistance metric · Hausdorff dimension · Spectral dimension

1 Introduction

The word "fractal" was coined by Benoit Mandelbrot in the year 1975. Formal mathematical definition of fractal states that a fractal is a set for which Hausdorff-Besicovitch dimension of an object strictly exceeds its topological dimension [1]. In general, a fractal is defined as a rough or fragmented geometric object that can be subdivided into parts, each of which is reduced- size copy of the whole [2]. Fractals like the von Koch curve and Sierpinski gasket are weakly described by their topological dimension [3]. Fractal objects are normally self-similar and independent of the scale [5]. Sierpinski discovered a set called the Sierpinski gasket (SG) at the

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beginning of this century [10]. Initially the physicists have started the analysis on fractal in the field of this ordered materials. The heat and water transfer in disorder materials such as polymers, fractured rocks, etc. are modeled into fractals. Related to this work, the mathematicians have developed the analysis on fractals in materials with irregular or fractal structures [2]. Laplacian on a fractal plays a vital role in analyzing materials with fractal structure. Laplacian operators converge to a refined operator under a proper scaling with dense domain, called the Laplacian on the Sierpinski gasket [6]. Kigami, J. has provided a general result that any Laplacian is regular corresponding to the effective resistance metric [7]. In this research paper, Laplacian and effective resistance metric on a finite Sierpinski fractal are discussed. The main idea for applying the effective resistance is so as to each Dirichlet form (Laplacian) over a finite fractal can be connected with an electrical network consists of resistors [4, 6]. He has proved that the effective resistance is shown metric with which the analytic properties of the fractal are discussed. Also he shows that the similarity dimension (S) of the fractal provides a useful intrinsic notion of dimension. This is the Hausdorff fractal dimension of the fractal object with respect to the effective resistance metric. The similarity dimension (S) is defined as the unique solution with respect to the resistance of the ith component. Kigami, J. has proved that for the post-critically finite self-similar fractal like Sierpinski gasket, the similarity dimension can be expressed in terms of the spectral dimension [6]. In Sect. 2, the construction of the Sierpinski gasket by the iterated function system is discussed. In Sect. 3, energy renormalization constant by Laplacian and then the scaling factor are obtained as the reciprocal of the energy renormalization constant and also the scaling constant is directly obtained. In Sect. 4, effective resistance metric and Laplacian on Sierpinski gasket are applied; thus Hausdorff fractal dimension and spectral fractal dimension are obtained.

2 Construction of Sierpinski Gasket

The construction of the Sierpinski gasket fractal starts with a filled-in equilateral triangle with sides of unit length, which is called as G_0. It is subdivided into four smaller triangles with side length $\frac{1}{2}$, by joining the midpoints of the sides, which are also equilateral triangles. As first iteration process the G_1 is obtained by removing the middle triangle which is rotated by 180° compared to other triangles. The boundary of that equilateral triangle is not removed. In each of the remaining three equilateral triangles, we remove the equilateral triangles formed by the midpoints of the three sides and so on. The set G_n contains 3^n triangles with side length 2^{-n}. Continuing this process, we get the Sierpinski gasket as the limiting case of the sequences $G_0, G_1, G_2, \ldots$ which is given in Fig. 1, and the Sierpinski Gasket is given as $G = \bigcap_{n=0}^{\infty} G_n$ [9].

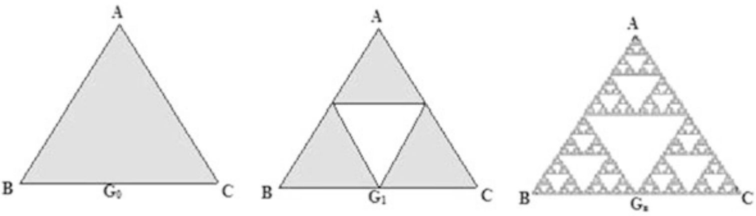

Fig. 1 The Sierpinski gasket

The iterated function system defined on the Sierpinski gasket is given by

$$F_i(x) = \frac{1}{2}(x - p_i) + p_i \ \text{ with fixed point } \ p_1, p_2, p_3.$$

This yields the following iterated function system. The IFS for the Sierpinski gasket is

$$f_1(x) = Ax$$

$$f_2(x) = Ax + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$f_3(x) = Ax + \begin{bmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{bmatrix} \text{ where } A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Next we apply the IFS on G_1, then we get G_2, and we apply the IFS process endlessly up to infinite times. The resultant graph is called a Sierpinski gasket, and it is the attractor for this IFS process and it is shown in Fig. 1.

3 Laplacian in Sierpinski Gasket

3.1 Laplacian in Finite Graphs [8]

The symmetry operator $H : l(S) \to l(S)$ which is linear and is known as a Laplacian operator on the set S and $l(S)$ is used for denoting the set of real-valued functions on the set S; then

$$l_0(V_n) = \{f \in l(V_n) : f(p) = 0 \ \text{ for } p \in V_0\} \tag{1}$$

For two sets S_1 and S_2, it is defined that

$$L(S_1, S_2) = \{H : l(S_1) \to l(S_2) \ \text{ and } H \text{ is linear}\}$$

In particular, $L(S)$ means $L(S, S)$.

3.2 *Vertex Degree* [3]

Let G be a simple graph consisting the set $V(G)$ of vertices and the set of $E(G)$ of edges. The two vertices a and b are adjacent vertices if they are associated by exactly one edge $e(a, b)$ in the simple graph.

3.3 *Standard and Normalized Laplacian*

Consider a function $f \in l(V(G))$, the graph (standard) Laplacian of the function f at any vertex x with Laplacian scaling constant $c(a, b)$, is defined by

$$\Delta f(a) = \sum_{e(a,b)\in E(G)} c(a, b)[f(a) - f(b)] \tag{2}$$

The notation $\hat{\Delta}$ denotes discrete Laplacian in a graph which is defined by

$$\hat{\Delta} f(a) = \frac{1}{\deg(a)} \sum_{e(a,b)\in E(G)} [f(a) - f(b)] \tag{3}$$

The matrix H is called the Laplacian matrix, which is symmetric and corresponds to Δ which is defined as

$$H_{i,j} = \begin{cases} 1 & \text{if } i \neq j \\ -\deg(a_i) & \text{if } i = j \text{ and } e(a_i, a_j) \in E(G) \\ 0 & \text{if otherwise} \end{cases} \tag{4}$$

for $x_i, x_j \in V(G)$.
The set (H_n, r) with weight r is the generalized standard Laplacian in the graph G_n, the nth iteration of the fractal graph. The matrix H_n is decomposed into

$$H_n = \begin{pmatrix} T_n & J_n^T \\ J_n & X_n \end{pmatrix} \tag{5}$$

Here $T_n \in L(G_0)$, $J_n \in L(G_0, G_n)$, and $X_n \in L(G_n)$, where G_0 and G_n are the initial and the nth iteration of the fractal graph, respectively. In particular $T = T_1$, $J = J_1$, and $X = X_1$.

Lemma 1 *The renormalization equation which relates the difference operator H_0 on G_0 with the difference operator H_1 on G_1 is defined by* [2]

$$\lambda H_0 = T - J^T X^{-1} J \ .$$

where λ is the renormalization constant.

Proof Let D be the Laplacian matrix on G_0. Then from Eq. (4), we get

$$H_0 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

The matrix corresponding to the standard Laplacian on G_1 is

$$H_1 = \begin{pmatrix} -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 1 & 1 \\ 1 & 0 & 1 & 1 & -4 & 1 \\ 1 & 1 & 0 & 1 & 1 & -4 \end{pmatrix}$$

Now from Eq. (5), by considering

$$H_1 = \begin{pmatrix} T & J^T \\ J & X \end{pmatrix}$$

$$T = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \; J = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \; X = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{pmatrix}$$

Applying the above matrices in the Lemma 1, we have

$$\begin{aligned} \lambda \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} - \begin{pmatrix} -0.8 & -0.6 & -0.6 \\ -0.6 & -0.8 & -0.6 \\ -0.6 & -0.6 & -0.8 \end{pmatrix} \\ &= \begin{pmatrix} -1.2 & 0.6 & 0.6 \\ 0.6 & -1.2 & 0.6 \\ 0.6 & 0.6 & -1.2 \end{pmatrix} \end{aligned}$$

Solving above matrices, we obtain $\lambda = \frac{3}{5}$, which is the renormalization constant of the Sierpinski gasket. Then H_0 together with the scaling $r = (1, 1, 1)$ is a harmonic structure. And the scaling constant is $c = \frac{1}{\lambda} = \frac{5}{3}$.

Lemma 2 $[N(1), N(2), N(3)]$ *is the gasket formed by the resistances* (r_1, r_2, r_3) *and corresponds with a self-similar energy with* $N = N(1) + N(2) + N(3)$*, where* c *is a constant such that* [10]

$$cr_1 = r_1 + \frac{N(2)N(3)}{N}(r_2 + r_3)$$
$$cr_2 = r_1 + \frac{N(1)N(3)}{N}(r_2 + r_3)$$
$$cr_3 = r_1 + \frac{N(1)N(2)}{N}(r_2 + r_3) \tag{6}$$

Proof For the Sierpinski gasket which is (1,1,1) gasket, and $r_1 = r_2 = r_3$ hence from Eq. (6), we obtain the scaling constant $c = \frac{5}{3}$.

4 Effective Resistance Metric and Laplacian in Sierpinski Gasket

The term effective resistance comes from electrical network analysis. The effective resistance $R(a, b)$ is defined as the resistance relating any two points a and b of the network after restricting that network to just those two points. The effective resistance relating any two vertices of a network or circuit is defined as the ratio of voltage across the nodes to the current flow injected into them. Consider the regular harmonic structure (D, r) on the Sierpinski gasket G and (ϵ, F). The intrinsic metric R between two points $a, b \in G$, called the effective resistance metric, is defined in terms of the Dirichlet form as

$$R(a, b) = [\min\{E(u, u) : u \in F, u(a) = 1, u(b) = 0\}]^{-1}$$

On the other hand, the effective resistance $R(a, b)$ is symbolized as the minimum value such that

$$|u(a) - u(b)|^2 \leq cE(u, u)$$

where the constant c represents the scaling factor of the Laplacian form. Using energy functions the effective resistance metric is defined by

$$R_{eff}(a, b) = \max\left\{\frac{(u(a) - u(b))^2}{\xi(u)};\ u \in Dom\xi \text{ and } \xi(u) > 0\right\} \tag{7}$$

where $\xi(u)$ is the energy of u, and it is defined by

$$\xi(u) = \lim_{m\to\infty} \xi(u)$$

Similarly for any two vertices u and v, it is defined as

$$\lim_{m\to\infty} \xi_m(u,u) = \xi(u,v)$$

The Hausdorff dimension D_H of the Sierpinski gasket fractal is the same as the similarity dimension (S), and then the similarity dimension (S) is described by the unique solution of the equation

$$\alpha = \left\{ s : \sum_{i=1}^{N} r_i^s = 1 \right\}$$

where r_i is the resistance of the ith component. And the sum of the components is denoted by N.

The resistance scaling constant (renormalization factor) is used in the computation of the Hausdorff fractal dimension which corresponds to the effective resistance metric. The Hausdorff fractal dimension of Sierpinski gasket corresponding to the effective resistance metric has been defined as

$$D_H = \frac{\log_e N}{\log_e c} \tag{8}$$

This Hausdorff fractal dimension is depending on the construction of Sierpinski gasket which means that it depends on the inner connection between the parts of Sierpinski gasket. For the existence of the limit, the spectral dimension of the Dirichlet form or Laplacian is defined by

$$D_S = \lim_{x\to\infty} \frac{2\log_e \lambda(x)}{\log_e x} \tag{9}$$

where the eigenvalue $\lambda(x)$ is the Laplacian counting function.
For standard Laplacian or Dirichlet form on Sierpinski gasket, spectral dimension can be calculated by using Hausdorff dimension value of the self-similar fractal:

$$D_S = \frac{2D_H}{D_H + 1} \tag{10}$$

Spectral dimension by the resistance scaling constant (renormalization factor) value is

$$D_S = \frac{2\log_e N}{\log_e (Nc)} \tag{11}$$

By Eq. (8), Hausdorff fractal dimension of Sierpinski Gasket which corresponds to the effective resistance metric is

$$D_H = \frac{\log_e 3}{\log_e(\frac{5}{3})} = \frac{\log_e 3}{\log_e 5 - \log_e 3} = 2.1507$$

This value of Hausdorff fractal dimension which corresponds to the effective resistance metric is different from Hausdorff fractal dimension of Sierpinski gasket which corresponds to the Euclidean metric. By Eq. (10), the spectral dimension of Sierpinski gasket using Hausdorff fractal dimension (D_H) of F which corresponds to the effective resistance metric R is

$$D_s(v) = \frac{2D_H}{D_H + 1} = \frac{2(2.1507)}{2.1507} = 1.3651$$

Now using the scaling constant of Sierpinski Gasket, $c = \frac{5}{3}$, in Eq. (11), the spectral dimension of Sierpinski gasket fractal is

$$D_S = \frac{2\log_e N}{\log_e(Nc)} = \frac{2\log_e 3}{\log_e 5} = 1.3651$$

The spectral fractal dimensions of Sierpinski gasket fractal obtained in both the cases are equal.

5 Conclusion

Here energy renormalization constant is calculated as the scaling factor by using Laplacian matrix method. Resistance scaling factor is obtained as the reciprocal of energy renormalization constant. Finally Hausdorff fractal dimension and spectral fractal dimension are obtained by effective resistance metric and Laplacian in Sierpinski gasket.

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