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PSEUDO-COMPLEMENTATION ON ALMOST DISTRIBUTIVE FUZZY LATTICES

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Abstract
In this paper, we introduce the concept of a Pseudo-Complementation* on an Almost Distributive Fuzzy Lattice (PCADFL) as a generalization of an almost Distributive Fuzzy Lattice (ADFL). It is proved that it is equationally definable on ADFL by using properties of Pseudo-Complementation on almost Distributive Lattice. We state and prove some results of a PCADFL, too.

Keywords: Almost Distributive Fuzzy Lattice (ADFL), Pseudo-Complementation, Fuzzy Partial Order Relation, Fuzzy Poset, Maximal Element, Pseudo-Complementation on Almost Distributive Fuzzy Lattice (PCADFL).

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INTRODUCTION

The general development of lattice theory started by G. Birkhoff [1]. The concept of an Almost distributive lattice (ADL) was introduced by U.M. Swamy and G.C. Rao [2] as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra. The structure of pseudo complemented distributive lattice I and II given by H. Lakser [8, 9] and G. Gratzer [9]. In [4] A. Berhanu, G. Yohannes and T. Bekalu introduced Almost distributive fuzzy lattice (ADFL). In [5] K. B. Lee proved that any Pseudo-complementation on a semi-lattice is equationally definable. The notion of Pseudo-complementation in an almost distributive lattices was introduced by U.M. Swamy, G.C. Rao and G.N. Rao in [3] and they observe that an almost distributive lattices have more than one pseudo-complementation while it is unique in case of distributive lattice. Pseudo-complements in semi-lattices introduced by O. Frink [6] and also by A.F, Lopez and M.I.T. Barrosa [7]. On the other hand, L.A. Zadeh [12] introduced Fuzzy sets to describe vagueness mathematically in its very abstractness and tried to solve such problems by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. In [13] N. Ajmal and K.V. Thomas defined a Fuzzy lattice as a fuzzy algebra and characterized fuzzy sublattices. I. Chon [14] considering the notion of fuzzy order of Zadeh, introduced a new notion of Fuzzy lattices and fuzzy partial order relations.

In this paper, we introduce the concept of Pseudo-Complementation $*$ on an ADFL and prove that it is equationally definable in ADFL. We characterized properties of Pseudo-Complementation on Almost distributive fuzzy lattice (PCADFL) and we give some preliminary results in PCADFL.

PRELIMINARIES

In this section, we recall certain elementary definitions and results required.

Definition 2.1. [2] An algebra $(\boldsymbol{R},\vee,\wedge,\boldsymbol{0})$ of type $(\boldsymbol{2},\boldsymbol{2},\boldsymbol{0})$ is called an Almost Distributive Lattice (ADL), if it satisfies the following axioms:

(L1) $\boldsymbol{a}\vee\boldsymbol{0}=\boldsymbol{a}$
(L2) $\boldsymbol{0}\wedge\boldsymbol{a}=\boldsymbol{0}$
(L3) $(\boldsymbol{a}\vee\boldsymbol{b})\wedge\boldsymbol{c}=(\boldsymbol{a}\wedge\boldsymbol{c})\vee(\boldsymbol{b}\wedge\boldsymbol{c})$
(L4) $\boldsymbol{a}\wedge(\boldsymbol{b}\vee\boldsymbol{c})=(\boldsymbol{a}\wedge\boldsymbol{b})\vee(\boldsymbol{a}\wedge\boldsymbol{c})$
(L5) $\boldsymbol{a}\vee(\boldsymbol{b}\wedge\boldsymbol{c})=(\boldsymbol{a}\vee\boldsymbol{b})\wedge(\boldsymbol{a}\vee\boldsymbol{c})$
(L6) $(\boldsymbol{a}\vee\boldsymbol{b})\wedge\boldsymbol{b}=\boldsymbol{b}$
For all $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in\boldsymbol{R}$
Now we give some basic results.
Lemma 2.2. [2] For any $\boldsymbol{a}\in\boldsymbol{R}$, we have
1. $\boldsymbol{a}\wedge\boldsymbol{0}=\boldsymbol{0}$
2. $\boldsymbol{a}\wedge\boldsymbol{a}=\boldsymbol{a}$
3. $\boldsymbol{a}\vee\boldsymbol{a}=\boldsymbol{a}$
4. $\boldsymbol{0}\vee\boldsymbol{a}=\boldsymbol{a}$.

Lemma 2.3. [2] For any $\boldsymbol{a},\boldsymbol{b}\in\boldsymbol{R}$, we have
1. $(\mathbf{a}\wedge\mathbf{b})\vee\mathbf{b}=\mathbf{b}$
2. $\mathbf{a}\vee(\mathbf{a}\wedge\mathbf{b})=\mathbf{a}=\mathbf{a}\wedge(\mathbf{a}\vee\mathbf{b})$
3. $\mathbf{a}\vee(\mathbf{b}\wedge\mathbf{a})=\mathbf{a}=(\mathbf{a}\vee\mathbf{b})\wedge\mathbf{a}$
4. $\mathbf{a}\vee\mathbf{b}=\mathbf{a}$ if and only if $\mathbf{a}\wedge\mathbf{b}=\mathbf{b}$
5. $\mathbf{a}\vee\mathbf{b}=\mathbf{b}$ if and only if $\mathbf{a}\wedge\mathbf{b}=\mathbf{a}$.

Definition 2.4. [2] For any $\boldsymbol{a},\boldsymbol{b}\in\boldsymbol{R}$, we say that $\boldsymbol{a}$ is less than or equal to $\boldsymbol{b}$ and write $\boldsymbol{a}\le\boldsymbol{b}$ is $\boldsymbol{a}\wedge\boldsymbol{b}=\boldsymbol{a}$ or equivalently, $\boldsymbol{a}\vee\boldsymbol{b}=\boldsymbol{b}$.
Lemma 2.5. [2] For any $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in\boldsymbol{R}$, we have
(1) $(\boldsymbol{a}\vee\boldsymbol{b})\wedge\boldsymbol{c}=(\boldsymbol{b}\vee\boldsymbol{a})\wedge\boldsymbol{c}$;
(2) $\wedge$ is associative in $\boldsymbol{R}$;
(3) $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}=\boldsymbol{b}\wedge\boldsymbol{a}\wedge\boldsymbol{c}$.
Definition 2.6. [4] Let $(\boldsymbol{R},\vee,\wedge,\boldsymbol{0})$ be an algebra of type $(\boldsymbol{2},\boldsymbol{2},\boldsymbol{0})$ and we call $(\boldsymbol{R},\boldsymbol{A})$ is an Almost Distributive Fuzzy Lattice(ADFL) if the following condition satisfied:
(F1) $\mathrm{A}(\mathrm{a},\mathrm{a}\vee 0)=\mathrm{A}(\mathrm{a}\vee 0,\mathrm{a})=1$
(F2) $\mathrm{A}(0,0\wedge\mathrm{a})=\mathrm{A}(0\wedge\mathrm{a},0)=1$
(F3) $\mathrm{A}\big((\mathrm{a}\vee\mathrm{b})\wedge\mathrm{c},(\mathrm{a}\wedge\mathrm{c})\vee(\mathrm{b}\wedge\mathrm{c})\big)$
$=\boldsymbol{A}\big((\boldsymbol{a}\wedge\boldsymbol{c})\vee(\boldsymbol{b}\wedge\boldsymbol{c}),(\boldsymbol{a}\vee\boldsymbol{b})\wedge\boldsymbol{c}\big)=\boldsymbol{1}$
(F4) $\mathrm{A}\big(\mathrm{a}\wedge(\mathrm{b}\vee\mathrm{c}),(\mathrm{a}\wedge\mathrm{b})\vee(\mathrm{a}\wedge\mathrm{c})\big)$
$\mathrm{A}\big((\mathrm{a}\wedge\mathrm{b})\vee(\mathrm{a}\wedge\mathrm{c}),\mathrm{a}\wedge(\mathrm{b}\vee\mathrm{c})\big)=1$
(F5) $\boldsymbol{A}\big(\boldsymbol{a}\vee(\boldsymbol{b}\wedge\boldsymbol{c}),(\boldsymbol{a}\vee\boldsymbol{b})\wedge(\boldsymbol{a}\vee\boldsymbol{c})\big)$
$=\boldsymbol{A}\big((\boldsymbol{a}\vee\boldsymbol{b})\wedge(\boldsymbol{a}\vee\boldsymbol{c}),\boldsymbol{a}\vee(\boldsymbol{b}\wedge\boldsymbol{c})\big)=\boldsymbol{1}$
(F6) $\boldsymbol{A}\big((\boldsymbol{a}\vee\boldsymbol{b})\wedge\boldsymbol{b},\boldsymbol{b}\big)=\boldsymbol{A}\big(\boldsymbol{b},(\boldsymbol{a}\vee\boldsymbol{b})\wedge\boldsymbol{b}\big)=\boldsymbol{1}$, for all $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in\boldsymbol{R}$.
Definition 2.7. [4] Let $(\boldsymbol{R},\boldsymbol{A})$ be an ADFL. Then for any $\boldsymbol{a},\boldsymbol{b}\in\boldsymbol{R}$, $\boldsymbol{a}\le\boldsymbol{b}$ if and only if $\boldsymbol{A}(\boldsymbol{a},\boldsymbol{b})>\boldsymbol{0}$.
Definition 2.8. [3] Let $(\boldsymbol{R},\vee,\wedge,\boldsymbol{0})$ be an ADL with 0. Then a unary operation $\boldsymbol{a}\to\boldsymbol{a}^*$on $\boldsymbol{R}$ is called a Pseudo-complementation on $\boldsymbol{R}$ if, for any $\boldsymbol{a},\boldsymbol{b}\in\boldsymbol{R}$, it satisfies the following conditions:
(P1) $a\wedge b=0\Longrightarrow a^*\wedge b=b$;
(P2) $a\wedge a^*=0$;
(P3) $(a\vee b)^*=a^*\wedge b^*$.

The unary operators $*$ is called Pseudo-complementation on $\boldsymbol{R}$.

Lemma 2.9. [3] Let $\boldsymbol{R}$ be an ADL with 0 and $*$ a pseudo-complementation on $\boldsymbol{R}$. Then, for any $\boldsymbol{a},\boldsymbol{b} \in \boldsymbol{R}$, we have the following:

1. 0^* is maximal;
2. if $\boldsymbol{a}$ is maximal, then $\boldsymbol{a}^* = \boldsymbol{0}$;
3. $0^{**} = \boldsymbol{0}$;
4. $\boldsymbol{a}^* \wedge \boldsymbol{a} = \boldsymbol{0}$;
5. $\boldsymbol{a}^{**} \wedge \boldsymbol{a} = \boldsymbol{a}$;
6. $\boldsymbol{a}^* = \boldsymbol{a}^{***}$;
7. $\boldsymbol{a}^* = \boldsymbol{0} \Leftrightarrow \boldsymbol{a}^{**}$ is maximal;
8. $\boldsymbol{a}^* \leq \boldsymbol{0}^*$;
9. $\boldsymbol{a}^* \wedge \boldsymbol{b}^* = \boldsymbol{b}^* \wedge \boldsymbol{a}^*$;
10. $\boldsymbol{a} \leq \boldsymbol{b} \Rightarrow \boldsymbol{b}^* \leq \boldsymbol{a}^*$;
11. $\boldsymbol{a}^* \leq (\boldsymbol{a} \wedge \boldsymbol{b})^*$ and $\boldsymbol{b}^* \leq (\boldsymbol{a} \wedge \boldsymbol{b})^*$;
12. $\boldsymbol{a}^* \leq \boldsymbol{b}^* \Leftrightarrow \boldsymbol{b}^{**} \leq \boldsymbol{a}^{**}$;
13. $\boldsymbol{a} = \boldsymbol{0} \Leftrightarrow \boldsymbol{a}^{**} = \boldsymbol{0}$.

Next, we give some properties and definitions of Fuzzy Partial Order Relation, Fuzzy Lattice and Fuzzy Distributive Lattice.

Definition 2.10. [14] Let $\boldsymbol{X}$ be a set. A function $\boldsymbol{A} : \boldsymbol{X} \times \boldsymbol{X} \rightarrow [\boldsymbol{0}, \boldsymbol{1}]$ is said to be fuzzy partial order relation if it satisfies the following condition:

1. $A(x,x) = 1$, for all x in X. That is A is reflexive.
2. $A(x,y) > 0$ and $A(y,x) > 0$ implies that $x = y$. That is A is antisymmetric.
3. $A(x,z) \geq \sup_{y \text{ in } x} \min [A(x,y), A(y,z)] > 0$. That is A is transitive.

If $\boldsymbol{A}$ is a fuzzy partial order relation in a set $\boldsymbol{X}$, then $(\boldsymbol{X},\boldsymbol{A})$ is a fuzzy partial order relation or fuzzy poset.

Definition 2.11. [14] Let $(\boldsymbol{X},\boldsymbol{A})$ be a fuzzy poset. Then $(\boldsymbol{X},\boldsymbol{A})$ is a fuzzy lattice if and only if $\boldsymbol{x} \vee \boldsymbol{y}$ and $\boldsymbol{x} \wedge \boldsymbol{y}$ exists for all $\boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}$.

Definition 2.12. [14] Let $(\boldsymbol{X},\boldsymbol{A})$ be a fuzzy lattice. Then $(\boldsymbol{X},\boldsymbol{A})$ is distributive if and only if $\boldsymbol{x} \wedge (\boldsymbol{y} \vee \boldsymbol{z}) = (\boldsymbol{x} \wedge \boldsymbol{y}) \vee (\boldsymbol{x} \wedge \boldsymbol{z})$ and $(\boldsymbol{x} \vee \boldsymbol{y}) \wedge (\boldsymbol{x} \vee \boldsymbol{z}) = \boldsymbol{x} \vee (\boldsymbol{y} \wedge \boldsymbol{z})$.

PSEUDO-COMPLEMENTATION ON ALMOST DISTRIBUTIVE FUZZY LATTICE

In this section, we give the definition of Pseudo-complementation on almost distributive fuzzy lattice (PCADFL) and develop some properties of a pseudo-complementation on ADL.

Definition 3.1. Let $(\boldsymbol{R}, \vee, \wedge, \boldsymbol{0})$ be an algebra of type $(\boldsymbol{2}, \boldsymbol{2}, \boldsymbol{0})$ and $(\boldsymbol{R},\boldsymbol{A})$ be a fuzzy poset. A unary operation $\boldsymbol{a} \longrightarrow \boldsymbol{a}^*$ on $\boldsymbol{R}$. Then $(\boldsymbol{R},\boldsymbol{A})$ is called a Pseudo-Complementation on Almost Distributive Fuzzy Lattice (PCADFL) for any $\boldsymbol{a},\boldsymbol{b} \in \boldsymbol{R}$, if the following conditions are satisfied:

(PF1) $A(1, a \vee b) = A(a \vee b, 1) = 1$
(PF2) $A(0, a \wedge b) = A(a \wedge b, 0) = 1$
(PF3) $A(a \wedge a^*, 0) = A(0, a \wedge a^*) = 1$
(PF4) $A(a^* \wedge b, b) = A(b, a^* \wedge b) = 1$
(PF5) $A((a \vee b)^*, (a^* \wedge b^*)) = A((a^* \wedge b^*), (a \vee b)^*) = 1$
(PF6) $A((a^*)^*, a) = A(a, (a^*)^*) = 1$

We can observe that the above six properties are independent.

Example 3.2: Let $(\boldsymbol{R},\boldsymbol{A})$ be an ADFL with 0 with at least two elements, if $\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{0} \wedge \boldsymbol{a}) > \boldsymbol{0}$ and $\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{0}^* \wedge \boldsymbol{a}) > \boldsymbol{0}$ then $\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{b}^*) > \boldsymbol{0}$ for all $\boldsymbol{a},\boldsymbol{b} \in \boldsymbol{R}$.

Example 3.3. Let $(\boldsymbol{R},\boldsymbol{A})$ be an ADFL with 0 and $(\boldsymbol{R}, +, . \boldsymbol{0})$ be a commutative regular ring. To each $\boldsymbol{a} \in \boldsymbol{R}$, let $\boldsymbol{a}^0$ be the unique idempotent element in $\boldsymbol{R}$ such that $\boldsymbol{aR} = \boldsymbol{a}^0\boldsymbol{R}$,

1. $A(a^0 b, a \wedge b) > 0$;
2. $A(a + (1 - a^0)b, a \vee b) > 0$;
3. $A(1 - a^0, a^*) > 0$;

for any $\boldsymbol{a},\boldsymbol{b} \in \boldsymbol{R}$, then $(\boldsymbol{R}, \vee, \wedge, \boldsymbol{0})$ is an ADFL with 0 and $*$ is a PCADFL on $(\boldsymbol{R},\boldsymbol{A})$.

Lemma 3.4. If $(\boldsymbol{R},\boldsymbol{A})$ be a PCADFL for each $\boldsymbol{a},\boldsymbol{b} \in \boldsymbol{R}$ then $\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{0}$ if and only if,

$\boldsymbol{A}(\boldsymbol{a} \wedge \boldsymbol{b}, \boldsymbol{0}) > \boldsymbol{0}$, $\boldsymbol{A}(\boldsymbol{a} \wedge \boldsymbol{b}, \boldsymbol{0}) > \boldsymbol{0}$, by antisymmetric property of $\boldsymbol{A}$.

Lemma 3.5. Let $(\boldsymbol{R},\boldsymbol{A})$ be an ADFL with $\boldsymbol{0}$ and $*$ on $(\boldsymbol{R},\boldsymbol{A})$ is a PCADFL, for any $\boldsymbol{a},\boldsymbol{b} \in \boldsymbol{R}$, then $\boldsymbol{A}(\boldsymbol{a}^* \wedge \boldsymbol{b}, \boldsymbol{b}) > \boldsymbol{0}$, the proof is trivial if and only if $\boldsymbol{a}^* \wedge \boldsymbol{b} = \boldsymbol{b}$.

Theorem 3.6. Let $(\boldsymbol{R},\boldsymbol{A})$ be a relatively complemented ADFL with $\boldsymbol{0}$ and with a maximal element $\boldsymbol{m}0$. Then $*$ be a PCADFL on $(\boldsymbol{R},\boldsymbol{A})$, for any $\boldsymbol{a},\boldsymbol{b} \in \boldsymbol{R}$. Then the following condition holds:

(1) $\boldsymbol{A}(\boldsymbol{b}, \boldsymbol{a}^* \wedge \boldsymbol{b}) = \boldsymbol{1}$
(2) $\boldsymbol{A}((\boldsymbol{a} \vee \boldsymbol{b}) \wedge \boldsymbol{a}^* \wedge \boldsymbol{b}^*, \boldsymbol{0}) = \boldsymbol{1}$
(3) $\boldsymbol{A}((\boldsymbol{a}^* \wedge \boldsymbol{b}^*) \vee (\boldsymbol{a} \vee \boldsymbol{b}), \boldsymbol{m}_0 \vee (\boldsymbol{a} \vee \boldsymbol{b})) = \boldsymbol{1}$

Proof. Let $(\boldsymbol{R},\boldsymbol{A})$ be an ADFL for any $\boldsymbol{a},\boldsymbol{b} \in \boldsymbol{R}$. Since $\boldsymbol{m}_0$ is maximal.

(1) $\boldsymbol{A}(\boldsymbol{b}, \boldsymbol{a}^* \wedge \boldsymbol{b}) = \boldsymbol{A}(\boldsymbol{b}, (\boldsymbol{a}^* \vee \boldsymbol{a}) \wedge \boldsymbol{b})$
$= \boldsymbol{A}(\boldsymbol{b}, (\boldsymbol{m}_0 \vee \boldsymbol{a}) \wedge \boldsymbol{b})$
$= \boldsymbol{A}(\boldsymbol{b}, \boldsymbol{m}_0 \wedge \boldsymbol{b})$
$= \boldsymbol{A}(\boldsymbol{b}, \boldsymbol{b})$
$= \boldsymbol{1}$. ***(since*** $(\boldsymbol{a}^* = \boldsymbol{a}^* \vee \boldsymbol{a})$***)***

Hence, $\boldsymbol{m}_0$ is maximal element $\boldsymbol{a} \vee \boldsymbol{a}^* = \boldsymbol{a} \vee \boldsymbol{m}_0$. Also, $(\boldsymbol{m}_0 \vee \boldsymbol{a} = \boldsymbol{m}_0)$ and $(\boldsymbol{m}_0 \wedge \boldsymbol{b} = \boldsymbol{b})$.

Therefore $\boldsymbol{A}(\boldsymbol{b}, \boldsymbol{a}^* \wedge \boldsymbol{b}) = \boldsymbol{1}$.

(2) $\boldsymbol{A}((\boldsymbol{a} \vee \boldsymbol{b}) \wedge \boldsymbol{a}^* \wedge \boldsymbol{b}^*, \boldsymbol{0})$
$= \boldsymbol{A}(((\boldsymbol{a} \vee \boldsymbol{b}) \wedge \boldsymbol{a}^*) \wedge ((\boldsymbol{a} \vee \boldsymbol{b}) \wedge \boldsymbol{b}^*), \boldsymbol{0})$ (by def 2.1.(3)
$= \boldsymbol{A}((\boldsymbol{a} \wedge \boldsymbol{a}^*) \vee (\boldsymbol{a}^* \wedge \boldsymbol{b}) \wedge (\boldsymbol{a} \wedge \boldsymbol{b}^*) \vee (\boldsymbol{b} \wedge \boldsymbol{b}^*), \boldsymbol{0})$
$= \boldsymbol{A}((\boldsymbol{a} \wedge \boldsymbol{m}_0) \vee (\boldsymbol{a}^* \wedge \boldsymbol{b}) \wedge (\boldsymbol{a} \wedge \boldsymbol{b}^*) \vee (\boldsymbol{b} \wedge \boldsymbol{m}_0), \boldsymbol{0})$
$= \boldsymbol{A}(((\boldsymbol{a} \wedge \boldsymbol{m}_0) \vee \boldsymbol{0}) \wedge (\boldsymbol{0}) \vee (\boldsymbol{b} \wedge \boldsymbol{m}_0), \boldsymbol{0})$
$= \boldsymbol{A}\big((\boldsymbol{a} \vee \boldsymbol{0}) \wedge (\boldsymbol{0} \vee \boldsymbol{b}), \boldsymbol{0}\big)$
$= \boldsymbol{A}(\boldsymbol{a} \wedge \boldsymbol{b}, \boldsymbol{0})$ (by def 3.1.(1))
$= \boldsymbol{A}(\boldsymbol{0}, \boldsymbol{0})$
$= \boldsymbol{1}$.

Since $\boldsymbol{m}_0$ is maximal element $\boldsymbol{a} \wedge \boldsymbol{a}^* = \boldsymbol{a} \wedge \boldsymbol{m}_0$. Also, from the definition of pseudo complementation on ADL $\boldsymbol{a}^* \wedge \boldsymbol{b} = \boldsymbol{0}$. Therefore $\boldsymbol{A}((\boldsymbol{a} \vee \boldsymbol{b}) \wedge \boldsymbol{a}^* \wedge \boldsymbol{b}^*, \boldsymbol{0} = \boldsymbol{1}$.

(3) $\boldsymbol{A}((\boldsymbol{a}^* \wedge \boldsymbol{b}^*) \vee (\boldsymbol{a} \vee \boldsymbol{b}), \boldsymbol{m}_0 \vee (\boldsymbol{a} \vee \boldsymbol{b}))$
$= \boldsymbol{A}((\boldsymbol{a}^* \vee (\boldsymbol{a} \vee \boldsymbol{b})) \wedge (\boldsymbol{b}^* \vee (\boldsymbol{a} \vee \boldsymbol{b})), \boldsymbol{m}_0 \vee (\boldsymbol{a} \vee \boldsymbol{b})$
$= \boldsymbol{A}(((\boldsymbol{a}^* \vee \boldsymbol{a}) \vee \boldsymbol{b}) \wedge ((\boldsymbol{b}^* \vee \boldsymbol{b}) \vee \boldsymbol{a}), \boldsymbol{m}_0 \vee (\boldsymbol{a} \vee \boldsymbol{b}))$
$= \boldsymbol{A}(((\boldsymbol{m}_0 \vee \boldsymbol{a}) \vee \boldsymbol{b}) \wedge ((\boldsymbol{m}_0 \vee \boldsymbol{b}) \vee \boldsymbol{a}), \boldsymbol{m}_0 \vee (\boldsymbol{a} \vee \boldsymbol{b}))$
$= \boldsymbol{A}((\boldsymbol{m}_0 \vee (\boldsymbol{a} \vee \boldsymbol{b})) \wedge (\boldsymbol{m}_0 \vee (\boldsymbol{a} \vee \boldsymbol{b})), \boldsymbol{m}_0 \vee (\boldsymbol{a} \vee \boldsymbol{b}))$
$= \boldsymbol{A}(\boldsymbol{m}_0 \vee (\boldsymbol{a} \vee \boldsymbol{b}), \boldsymbol{m}_0 \vee (\boldsymbol{a} \vee \boldsymbol{b})$
$= \boldsymbol{1}$.

Since $\boldsymbol{m}_0$ is maximal element and also by the properties of ADL $\boldsymbol{a} \wedge \boldsymbol{a} = \boldsymbol{a}$.

$\therefore$ $\boldsymbol{A}((\boldsymbol{a}^* \wedge \boldsymbol{b}^*) \vee (\boldsymbol{a} \vee \boldsymbol{b}), \boldsymbol{m}_0 \vee (\boldsymbol{a} \vee \boldsymbol{b})) = \boldsymbol{1}$. Hence $*$ is PCADFL on $(\boldsymbol{R},\boldsymbol{A})$.

Now, we give some properties of PCADFL in the following lemma.

Lemma 3.7. Let $(\boldsymbol{R},\boldsymbol{A})$ be an ADFL with $\boldsymbol{0}$ and $*$ be a PCADFL on $\boldsymbol{R}$.

Then, for any $\boldsymbol{a},\boldsymbol{b} \in \boldsymbol{R}$, we have the following:

(1) $\boldsymbol{0}^*$ is maximal;
(2) if $\boldsymbol{a}$ is maximal, then $\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{a}^*) = \boldsymbol{1}$;
(3) $\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{0}^{**}) = \boldsymbol{1}$;
(4) $\boldsymbol{A}(\boldsymbol{a}^* \wedge \boldsymbol{a}, \boldsymbol{0}) = \boldsymbol{1}$;
(5) $\boldsymbol{A}(\boldsymbol{a}, \boldsymbol{a}^{**} \wedge \boldsymbol{a}) = \boldsymbol{1}$;
(6) $\boldsymbol{A}(\boldsymbol{b}^{***}, \boldsymbol{b}^*) = \boldsymbol{1}$;
(7) $\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{a}^*) > \boldsymbol{0} \Leftrightarrow \boldsymbol{a}^{**}$ is maximal;
(8) $\boldsymbol{A}(\boldsymbol{a}^*, \boldsymbol{0}^*) > \boldsymbol{0}$;
(9) $\boldsymbol{A}(\boldsymbol{b}^* \wedge \boldsymbol{a}^*, \boldsymbol{a}^* \wedge \boldsymbol{b}^*) = \boldsymbol{1}$;
(10) $\boldsymbol{A}(\boldsymbol{b}^*, \boldsymbol{a}^*) > \boldsymbol{0}$;
(11) $\boldsymbol{A}(\boldsymbol{a}^*, (\boldsymbol{a} \wedge \boldsymbol{b})^*) > \boldsymbol{0}$ and $\boldsymbol{A}(\boldsymbol{b}^*, (\boldsymbol{a} \wedge \boldsymbol{b})^*) > \boldsymbol{0}$;
(12) $\boldsymbol{A}(\boldsymbol{b}^{**}, \boldsymbol{a}^{**}) > \boldsymbol{0}$;
(13) $\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{a}^{**}) > \boldsymbol{0}$.

Proof. (1) For any $\boldsymbol{a} \in \boldsymbol{R}, \boldsymbol{A}(\boldsymbol{0}, \boldsymbol{0} \wedge \boldsymbol{a}) > \boldsymbol{0}$ and hence, $\boldsymbol{A}(\boldsymbol{a}, \boldsymbol{0}^* \wedge \boldsymbol{a}) > \boldsymbol{0}$ which implies that $\boldsymbol{0}^*$ is maximal.

(2) Suppose $\boldsymbol{a}$ is maximal

$\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{a}^*) = \boldsymbol{A}(\boldsymbol{0}, (\boldsymbol{a} \vee \boldsymbol{a}^*)^*)$ (since $\boldsymbol{a} = \boldsymbol{a} \vee \boldsymbol{a}^*$))
$\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{a}^*) = \boldsymbol{A}(\boldsymbol{0}, (\boldsymbol{a} \vee \boldsymbol{a}^*)^*)$ (since $\boldsymbol{a} = \boldsymbol{a} \vee \boldsymbol{a}^*$))
$= \boldsymbol{A}(\boldsymbol{0}, \boldsymbol{a}^* \wedge \boldsymbol{a}^{**})$ (since $\boldsymbol{a}^* \wedge \boldsymbol{a}^{**} = \boldsymbol{0}$)
$= \boldsymbol{A}(\boldsymbol{0}, \boldsymbol{0})$
$= \boldsymbol{1}$.

Therefore $\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{a}^*) > \boldsymbol{0}$ if and only if $\boldsymbol{a}$ is maximal.

(3) Follows from (1) and (2) $\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{0}^{**}) = \boldsymbol{1}$.

(4) Follows from definition of PCADFL $\boldsymbol{A}(\boldsymbol{a}^* \wedge \boldsymbol{a}, \boldsymbol{0}) = \boldsymbol{1}$.

(5) Since $\boldsymbol{a} \wedge \boldsymbol{a}^* = \boldsymbol{0}$ qwhich implies $\boldsymbol{a}^* \wedge \boldsymbol{a} = \boldsymbol{0} \Rightarrow \boldsymbol{a}^{**} \wedge \boldsymbol{a} = \boldsymbol{a}$ (by Lemma 2.9.(5)).

Hence, $\boldsymbol{a} \leq \boldsymbol{a}^{**} \wedge \boldsymbol{a}$ if and only if $\boldsymbol{A}(\boldsymbol{a}, \boldsymbol{a}^{**} \wedge \boldsymbol{a}) > \boldsymbol{0}$ by absorption law of ADFL.

(6) $A(b^{***}, b^*)$
$= A((b^{**} \vee b)^*, b^*)$ (since $b^{**} = b^{**} \vee b$)
$= A(b^{***} \wedge b^*, b^*)$ (since $b^{***} \wedge b^* = b^*$)
$= A(b^*, b^*)$
$= 1$.
Therefore $A(b^{***}, b^*) = 1$.
(7) Follows from (1), (2) and (6) $A(0, a^*) > 0$ for any $a \in R$ if and only if a^{**} is maximal.
(8) $A(a^*, 0^*) = A((a \vee a^*)^*, 0^*)$ (from (2)
$= A(a^* \wedge a^{**}, 0^*)$
$= A(0^*, 0^*)$ (since $a^* \wedge a^{**} = 0^*$)
$= 1 > 0$.
Therefore $a^* \leq 0^*$ which implies $A(a^*, 0^*) > 0$.
(9) $A((b^* \wedge a^*), (a^* \wedge b^*))$
$= A((b \vee a)^*, (a^* \wedge b^*))$ (by def 2.8.(3))
$= A((a \vee b)^*, (a^* \wedge b^*))$(by lemma 2.5.(1))
$= A(a^* \wedge b^*, a^* \wedge b^*)$
$= 1$.
Therefore $A(b^* \wedge a^*, a^* \wedge b^*) = 1$.
(10) $A(b^*, a^*) = A(a \vee b)^*, a^*)$ (since $b = a \vee b$)
$= A(a^* \wedge b^*, a^*)$ (by def 2.8.)
$= A(a^*, a^*) > 0$.
Hence, $a \leq b \Leftrightarrow b^* \leq a^*$ if and only if $A(b^*, a^*) > 0$ by antisymmetric property of A.
(11) Hence $a^* \leq (a \wedge b)^*$ which implies $A(a^*, (a \wedge b)^*) > 0$ and similarly we have $a \wedge b \leq b$ which implies that $b^* \leq (a \wedge b)^*$ if and only if $A(b^*, (a \wedge b)^*) > 0$ by antisymmetric property of A for each $a, b \in$ R.
(12) Follows from (6) and (10), $A(a^*, b^*) > 0$ which implies $A(b^{**}, a^{**}) > 0$.
(13) For any $a \in R$, follows from (3) if and only if a^{**} is maximal. Hence $a^{**} \leq 0$ which implies that $A(a^{**}, 0) > 0$.
More generally, we have
Lemma 3.8. Let (R, A) be an ADFL with 0 and $*$ be a PCADFL on (R, A). Then, for any $a, b \in R$, the following holds:
(14) $A(0, a \wedge b) > 0$ and $A(a \wedge b, 0) > 0$;
(15) $A(0, a^{**} \wedge b) > 0$ and $A(a^{**} \wedge b, 0) > 0$;
(16) $A(0, a^{**} \wedge b^{**}) > 0$ and $A(a^{**} \wedge b^{**}, 0) > 0$;
(17) $A(0, a \wedge b^{**}) > 0$and $A(a \wedge b^{**}, 0) > 0$.
Using the absorption laws of ADFL, $*$ satisfies the given equations of PCADFL on (R, A).
Lemma 3.9. Let (R, A) be an ADFL with 0 and $*$ is a PCADFL on (R, A). Then, for any $a, b \in R$, the following holds:
(1) $A((a \wedge b)^{**}, a^{**} \wedge b^{**}) = 1$
(2) $A((a \wedge b)^*, (b \wedge a)^*) = 1$
(3) $A((a \vee b)^*, (b \vee a)^*) = 1$
Proof. Let $a, b \in R$.
(1) $A((a \wedge b)^{**}, a^{**} \wedge b^{**})$
$= A((a \wedge b)^{**}, (a^* \vee b^*)^*)$
$= A((a \wedge b)^{**}, (a \wedge b)^{**})$
$= 1$.
Therefore $A((a \wedge b)^{**}, a^{**} \wedge b^{**}) = 1$.
(2) $A((a \wedge b)^*, (b \wedge a)^*)$
$= A((a^* \vee b^*), (b \wedge a)^*)$
$= A((b^* \vee a^*), (b \wedge a)^*)$
$= A(b \wedge a)^*, (b \wedge a)^*)$
$= 1$.
Therefore $A((a \wedge b)^*, (b \wedge a)^*) = 1$.
(3) $A((a \vee b)^*, (b \vee a)^*)$
$= A((a \vee b)^*, (b^* \wedge a^*))$ (by def 2.8.(3))
$= A((a \vee b)^*, (a^* \wedge b^*))$ (by lemma 2.5.(1))
$= A((a \vee b)^*, (a \vee b)^*)$
$= 1$.
Therefore $A((a \vee b)^*, (b \vee a)^*) = 1$.
Theorem 3.10. Let $(R, \vee, \wedge, 0)$ be an ADFL with 0. Then a unary operation $*$ is a PCADFL on (R, A), if and only if the following equations are satisfied:

1. $A(b \wedge b^*, 0) > 0$;
2. $A(b^{**} \vee b), b^{**}) > 0$;
3. $A((b \vee a)^*, b^* \wedge a^*) > 0$;
4. $A((b \wedge a)^{**}, b^{**} \wedge a^{**}) > 0$.

Proof. Let $a, b \in R$. Suppose $A(a, b) > 0$ which implies $a \leq b$, then (R, A) be an ADFL.
(1) For any $(a, b) \in R$, whenever $b \wedge b^* \leq 0$ implies that $A(b \wedge b^*, \;\; 0) > 0$ by antisymmetric property of A. Since $b \wedge b^* = 0$.Hence $A(b \wedge b^*, \;\; 0) > 0$.
(2) $A(b^{**} \vee b), b^{**}) = A(b^{**}, b^{**})$
$= 1 > 0$.
Since b^{**} is the maximal element, whenever $b^{**} \vee b \leq b^{**}$ implies that $A(b^{**} \vee b, b^{**}) > 0$.
(3) $A((b \vee a)^*, b^* \wedge a^*)$
$= A((b^* \wedge a^*), (b^* \wedge a^*))$
$= 1 > 0$.
Hence $(b \vee a)^* \leq b^* \wedge a^*$ implies that $A((b \vee a)^*, b^* \wedge a^*) > 0$ and $A((b^* \wedge a^*), (b \vee a)^* > 0$by antisymmetric property of A.
(4) $A((b \wedge a)^{**}, b^{**} \wedge a^{**})$
$= A((b \wedge a)^{**}, (b^* \vee a^*)^*)$ (by 3.9.(2))
$= A((b \wedge a)^{**}, (b \wedge a)^{**})$
$= 1 > 0$.
Therefore $A((b \wedge a)^{**}, b^{**} \wedge a^{**}) > 0$. Hence, (R, A) is a PCADFL for any $(a, b) \in R$.
Theorem 3.11. Let $(R, \vee, \wedge, 0)$ be an ADFL. Then a unary operation $*$ is a PCADFL on (R, A), if and only if, the following equations are satisfied:

1. $A((a \wedge b)^* \wedge b, a^* \wedge b) = 1$
2. $A(a, 0^* \wedge a) = 1$
3. $A(0^{**}, 0) = 1$
4. $A((a \vee b)^*, (a^* \wedge b^*)) = 1$
5. $A(a \wedge b \wedge (a \wedge b)^*, 0) = 1$

Proof. Let (R, A) is a ADFL for $a, b \in R$, whenever $A(a, b) > 0$.
(1) $A((a \wedge b)^* \wedge b, a^* \wedge b)$
$= A((a^* \vee b^*) \wedge b, a^* \wedge b)$
$= A((a^* \wedge b) \vee (b^* \wedge b), a^* \wedge b)$
$= A((a^* \wedge b) \vee (b \wedge b^*), a^* \wedge b)$
$= A((a^* \wedge b) \vee 0), a^* \wedge b)$ (by def 2.8.(2))
$= A((a^* \vee 0) \wedge (b \vee 0), a^* \wedge b)$
$= A(a^* \wedge b, a^* \wedge b)$
$= 1$.
from the definition of pseudo-complemented on ADL $(a \wedge b)^* = a^* \vee b^*$ and $b \wedge b^*$=0.
Therefore $A((a \wedge b)^* \wedge b, a^* \wedge b) = 1$.
(2) $A(a, 0^* \wedge a) = A(a, 1 \wedge a)$ (since $0^* = 1$)
$= A(a, a)$
$= 1$.
Therefore $A(a, 0^* \wedge a) = 1$.
Hence $A(a, 0^* \wedge a) > 0$. For $a \in R$.
(3) $A(0^{**}, 0) = A(0,0)$ (by lemma 2.9.(3))
=1.
Therefore $A(0^{**}, 0)$= 1.
(4) $A((a \vee b)^*, (a^* \wedge b^*))$
$= A((a^* \wedge b^*), (a^* \wedge b^*))$
=1.
Therefore $((a \vee b)^*, (a^* \wedge b^*) = 1$, which implies that $A((a \vee b)^*, (a^* \wedge b^*)) > 0$ by antisymmetric property of A.
(5) $A(a \wedge b \wedge (a \wedge b)^*, 0)$
$= A(a \wedge b \wedge (a^* \vee b^*), 0)$ (by lemma 3.9.(2))
$= A(0 \wedge (a^* \vee b^*), 0)$ (by def 2.8.(1))
$= A((0 \wedge a^*) \vee (0 \wedge b^*), 0)$
$= A\big((0) \vee (0), 0\big)$
$= A(0, 0)$
$= 1$.
Hence $A(a \wedge b \wedge (a \wedge b)^*, 0) = 1$. Similarly$A(0, a \wedge b \wedge (a \wedge b)^*) = 1$. for$a, b \in R$, Therefore $A(a \wedge b \wedge (a \wedge b)^*, 0) = 1$, which implies that $A(a \wedge b \wedge (a \wedge b)^*, \;\; 0) > 0$ by antisymmetric property of A. Hence $*$ is a pseudo-complement on R and (R, A) be a PCADFL.

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