Mathematics and Statistics 11(5): 816-819, 2023
DOI: 10.13189/ms.2023.110507 http://www.hrpub.org

MID-units in Right Duo-seminearrings

S. Senthil[1,2], **R. Perumal**[2,*]

[1]Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies,India

[2]Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, India

Received January 13, 2023; Revised June 22, 2023; Accepted July 10, 2023

Cite This Paper in the following Citation Styles

***(a)**: [1] S. Senthil, R. Perumal, "MID-units in Right Duo-seminearrings," Mathematics and Statistics Vol.11, No.5, pp. 816-819, 2023. DOI: 10.13189/ms.2023.110507*

***(b)**: S. Senthil, R. Perumal, (2023). MID-units in Right Duo-seminearrings. Mathematics and Statistics, 11(5), 816-819. DOI: 10.13189/ms.2023.110507*

Abstract In this paper, we focus on a subclass of duo-seminearrings called as right duo-seminearrings. We also focus on the algebraic properties and peculiarities of mid-units within this class. As a logical extension of the concept of mid-identities in semirings, the concept of mid-units in right pair seminearings is introduced. Mid-units are elements with both left and right invertibility, making them essential for understanding the structure and behaviour of right duo-seminearrings. In particular, we examine the interaction between idempotents and seminearring mid-units. We have also investigated regular right duo-seminearring which is a semilattice of subseminearrings with mid-units. In order to have a mid-unit in duo-seminearrings, we have established the necessary and sufficient conditions. The aim of this work is to carry out an extensive study on algebraic structure of right duo-seminearrings and the major objective is to further enhance the theory of right duo-seminearrings in order to find special structures of right duo-seminearrings. Throughout the research, rigorous proofs are provided to support the theoretical developments and ensure the validity of the findings. Concrete examples are also presented to illustrate the concepts and facilitate a better understanding of the algebraic structures associated with duo seminearings and mid-units. These examples serve as valuable tools for researchers and practitioners interested in the application of right duo-seminearrings and mid-units in their respective fields. Due to their applicability in domains such as computer science, cryptography, and coding theory, the topic of duo seminearrings, which generalise both semirings and duo-rings, have received substantial attention in algebraic research.

Keywords Right Duo, MID-units, Seminearring, Sub-seminearring, Idempotent

1 Introduction

Seminearring is an algebraic structure from monoids that was first developed by Hoorn and Rootselaar in 1967 [1]. The binary operations such as addition and multiplication are referred to as a right seminearring $(S, +, \cdot)$ in right-distributive semigroups $(S, +)$ and $(S, \cdot)$ [2]. If $0 + x = x + 0 = x$ and $0.x = x.0 = 0$ for every $x \in S$, then S is an absorbing zero '0'. The seminearring applications include linear sequential machines, semigroup mapping, etc. An absorbing zero semigroup mapping set is $(\Gamma, +)$ with respect to $\mathfrak{M}(\Gamma)$ being the composition of mapping and the form of pointwise addition being the fundamental instance of seminearrings.

Yamada introduced the notion of mid-unit semigroups [3]. Ault [4] and Blyth [5] have done researches on the characteristics of the mid-units of regular semigroups. The notion of right duo-seminearrings is introduced and verified when S is a regular seminearring. In addition, some of the distinguishing characteristics of a right duo-seminearring are also proved [9]. The concept of mid-units in right duo seminearrings is extended in this study and has presented some findings related to mid-units in seminearrings. The paper is organized in such a way that the basic concepts are given in section 2. Section 3 examines the characterization of seminearrings' mid-units in further detail. In section 4, several significant findings of mid-units on the right duo-seminearrings were investigated. In section 5 the duo seminearrings with mid-units are highlighted and concluded in section 6.

2 Preliminaries

Definition 2.1 [1]. A right seminearring is a system $(S, \cdot, +)$ then
(i) $(S,\ \cdot)$ is a semigroup
(ii) $(S,\ +)$ is a semigroup
(iii) $(x_1 + x_2)\, x_3 =\ x_1x_3 + x_2x_3$ for all $x_1,\ x_2,\ x_3 \in S$.

(iv) There exists $0 \in S$ such that $0 + x = x + 0 = s$ for all $x \in S$.
(v) $0x = 0$ for every $x \in S$.

Every duo seminearrings used in this work is a right duo seminearrings.

Definition 2.2 [11] If a seminearring S has a non-empty subset I which satisfies the following criteria, then A strong ideal of S is said to be I :
(i) For $a, b \in I, a + b \in I$ $(I + I \subseteq I)$.
(ii) For $x \in S,\ x + I \subseteq I + x$.
(iii) If $a \equiv 0$ then $x \in I$.
(iv) $Ix \subseteq I$ for all $x \in S$ (right strong ideal).
(v) $x(I + x') \subseteq I + xx'$ for every $x,\ x' \in S$ (left strong ideal).

Theorem 2.3 [11] (i) If I and J are strong ideals of a S then $I \cap J$ is a strong ideal of J and $(I + J)/I \cong J/(I \cap J)$.
(ii) If I and J are strong ideals of a seminearring S and $I \subseteq J$ then $S/J \cong (S/I)/(J/I)$.

Definition 2.4 [12] $Sc = \{x \in S \mid xx' = x,$ for all $x' \in S\}$ is referred as a constant part of a seminearring S. $S_0 = \{x \in S \mid s0 = 0\}$ is known as a zero-symmetric part of a seminearring S.

Definition 2.5 [13] A subsemigroup is defined as a non-empty subset M of a semigroup $(S, +)$, if $x,\ y \in M$ implies $x + y \in M$.

Definition 2.6 [12] An additive subsemigroup M of a seminearring S is referred as a subseminearring, if $0 \in M$ and $MM \subseteq M$.

Definition 2.7 [7] If $MS \subseteq M$ and $SM \subseteq M$, a subseminearring M of a seminearring S is said to be invariant.

For the results related to lattices, we refer to [14] and for boolean nearrings we refer to [15].

Definition 2.8 [8] If a class σ of seminearrings $A/B \in \sigma$, where A is a left invariant strong ideal of S, and B is a strong ideal of S, then condition (F_1) for that class of seminearrings is said to be satisfied for each strong ideal B of A,

Definition 2.9 [10] If any ideal of S is idempotent, then S is a semisimple object. If $s = sts$ for each $s \in S$, $\exists\, t \in S$, then S is regular. If $a \in a^2\ S(Sa^2)\ \forall\, a \in S$, therefore S is a right (left) regular seminearring.

Definition 2.10 [6] If the non-empty subset A of S is
(i) $C(A) = \{x \in S/xa = ax\ \forall\, a \in A\}$,
(ii) $C(S) = \{x \in S/xa = ax\ \forall\, a \in S\}$ is referred as the center of S, when $A = S$.
(iii) When $E \subseteq C(S)$, then it is said to be central idempotents.

For a given $a \in S$, it is idempotent if $a^2 = a$.

Definition 2.11 [6] If every $a \in S$ is idempotent, such that S is a band. It is referred to as a rectangular band if it is a band and $xyz = xz\ \forall\, x,\ y,\ z \in S$. If $r = s$ is implied for all $r,\ s \in S,\ r^2 = rs = s^2$ then S is a separative seminearring.

Definition 2.12 [6] A relation of equivalence, if ρ is both a left and a right compatible function, $x\rho y \implies rx\rho ry$ and $xr\rho yr$, for all $r \in S$, then ρ on S is considered to be a congruence. If for idempotents $e,\ f \in S,\ e\rho f \implies e = f$ then a congruence ρ on S is known as separative idempotent.

3 Seminearrings with Mid-Units

Definition 3.1. For all $a, b \in S$, an element $u \in S$ meeting the conditions of $bua = ba$ is referred to as a seminearring S with mid-units.

$$bua = ba,\ \forall\, a,\ b\ \in S$$

We first observe that if any S has an identity, it is unable to have another mid-units. All elements of a rectangular band and a left(right) zero semigroup are mid-units. Instead of a unit an idempotent does not have to be a mid-unit and a mid-unit does not have to be an idempotent, as shown in the example below.

Example 3.2. Analyze the addition and multiplication table below with the binary operations in seminearring $S = \{0, a, b, c, d\}$.

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	b	b
d	0	a	d	d	d

The elements $b,\ c$, and d are mid-units in S. The idempotent a is not a mid-unit, and c is not an idempotent. In the follow-up, M represents the set of all the mid-units and E represents the set of all the idempotents of a S.

Proposition 3.3. Assume S with $M \neq \emptyset$. In reality, $E \cap M = M^2$ and $E \neq \emptyset$ follows. M^2 is also a rectangular band. Therefore, S has a basic subseminearring. Additionally, each component of M^2 is a maximum idempotent of S.

Proof. Let $u, v \in M$. Such that $uv \in M$. For, $a\,(uv)\,b = vb\,(au) = vba = ba\ \ \forall\ a,\ b \in S$. Thus $M^2 \subseteq M$. Also $uvuv = u\,(vu) = uv$ and so $M^2 \subseteq E$. Hence $M^2 \subseteq M \cap E$. If $u \in M \cap E$, then $u = u^2 \in M^2$, so that $M \cap E = M^2$. Further M^2 is a rectangular band, since
$vwu = wu$, for all $u, v, w \in M^2$.
Suppose $m \in M^2$ when $m \leq e$. For any $e \in E$. Hence $m = me = em = e\,(em) = e^2 = e$, establishing that m in E is maximal.

Definition 3.4. An element a of S is termed as right(left) $E^* -$ regular, if there exists $e \in E\ \ (f \in E)$ clearly $a = ae\,(a = fa)$. S is referred as $E^* -$ regular seminearring if every $a \in S$ is right(left) $E^* -$ regular.

Any regular seminearring is obviously a right(left) $E^* -$ regular seminearring. The converse is not true, hence the seminearring $S = \{1,\ a,\ a^2,\ \dots\}$.

Theorem 3.5. A seminearring S with mid-unit u, is an idempotent iff it is right (left) $E^* -$ regular.

Proof. It is obvious that if u is idempotent, then it is right $E^* -$ regular

In contrast, suppose u be right $E^* -$ regular. Such that $u = ue$, for some $e \in E$. Hence $u^2 = u\,(ue) = ue = u$, showing that u is an idempotent. A similar proof holds in the case when 'right' is replaced by 'left'.

Corollary 3.6. A regular seminearring has idempotent mid-units.

More generally, we have

Proposition 3.7. A right sided regular seminearring has idempotent mid-units.

Proposition 3.8. In a seminearring S, a mid-unit is a central idempotent m iff $S^2 = mS$.

Proof. Suppose m be the mid-unit in S and let $x \in S^2$ such that

$$x = ab\,(a,\ b\ \in S)$$

$$= bma = mab \in mS$$

Therefore $S^2 \subseteq mS$. But $mS \subseteq S^2$, clearly that $S^2 = mS$.
Conversely, suppose $S^2 = mS$. If $a,\ b \in S$, such that $ab = mq,$ for any $q \in S$ and hence

$$bma = mab = m(qm),$$

$$= mq = ab.$$

Therefore, m is the mid-unit in S.

Corollary 3.9. A seminearring's mid-unit has a central idempotent which is unique.

Proof. Assume that the mid-unit of central idempotent S are $u,\ v$.
According to Proposition 3.9, $S^2 = uS$ and $S^2 = vS$. Since $u = u^2 \in S^2 = vS$, we have $u = vq,\ q \in S$. Hence

$$vu = vvq = vq = u.$$

Similarly

$$uv = u.$$

$$v = vu = uv = u.$$

Proposition 3.10. The only mid-unit of an inverse seminearring is the identity.

Proof. Let S be an inverse seminearring and u, a mid-unit of S.
S being regular, u is an idempotent (by corollary 3.7). Let $a \in S$. Then

$$ua = uaa^{-1}a$$

$$= u\left(a^{-1}a\right)a$$

$$= ua\left(a^{-1}a\right)$$

as $a^{-1}a$ is an idempotent and idempotents commute in S. Thus $ua = uaa^{-1} = a$. Similarly $au = a$ and this proves the result.

4 Right duo with mid-units-seminearrings

Regular right duo-seminarrings with mid-units are discussed in this section.

Definition 4.1. S is defined as
(i) if every one-sided ideal of S has two sides, then it is considered as duo.
(ii) if every right ideal of S has two-sides, then it is right duo.

Proposition 4.2. A regular right duo-seminearring is formed by the mid-units of S.

Proof. The set M of mid-units of S is undoubtedly a seminearring itself. Considering that each $e \in M$ is an idempotent, M is also regular (by corollary 3.7). Furthermore, if $e, f \in M$, such that $ef = xe$, where $x \in S^1$, so that $efe = fe$. Thus $fe = efe = e^2 = e, f$ as a mid-units. As a result M is a right zero seminearring. Therefore M is right duo, hence the proof.

In view of [10], a regular right duo-seminearring is orthodox and hence [5] gives rise to the following structure theorem.

Theorem 4.3. A regular right duo-seminearring S with a mid-unit u is a $F-$ spined product of uS and Su subsemigroups.

The following example shows that there exist regular right duo-seminearrings without mid-units.

Example 4.4. The seminearring $S = \{a,\ b,\ c\}$ with the following multiplication table is regular and duo. But none of $a,\ b, c$ is a mid-unit of S.

$\cdot$	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c

However we prove the following theorem for subseminearrings with mid-units.

Theorem 4.5. Any regular right duo-seminearring S is a semilattice of subseminearrings with mid-units.

Proof. Since S is regular, it is a semilattice of subseminearrings S_α, where each S_α is a rectangular group and so the idempotents of S_α are the mid-units of S_α. This completes the proof.

5 Mid-Units in Duo-Seminearrings

Proposition 5.1. In S, the following holds.

1. M is the center of S and represents the distinct maximum idempotent of S.
2. $Ma = aM = \{au\}$ for every $a \in S$ and $u \in M$.

Proof.

1. Let $u \in M$. Such that $u^2 \in M^2 = M \cap E$. By Proposition 3.4, u^2 is a maximum idempotent of S.

Now, as u^2 is an idempotent, according to Proposition 4.2. it is central in S. Clearly Corollary 3.10 demonstrates that the idempotent in M^2 is unique. Considering any $a \in S$ and $u \in M$ to show that M is central. Such that
$au = auu = au^2 = u^2a$ (u^2 being central)
=ua
This proves (1).

1. Let $v \in M$ be arbitrary. Then

$va = vua$, for each $u \in M$
$= uva$ (by (1))
$= ua$ and so.
$Ma = aM = ua$.
This proves the result.

Corollary 5.2 If S is separative then for all $a, b\ \in S$ $a^2 = ab = b^2$ implies $a = b$. Clearly the mid-unit is unique.

Proof. According to Proposition 5.1 (1), $u^2 = uv = v^2$ and $u = v$ follows if $u,\ v$ are two mid-units of S.

Next, we propose a necessary and sufficient conditions for the duo-seminearring of mid-units.

Theorem 5.3. If and only if S^2 has identity, S is said to have a mid-unit.

Proof. Suppose u be the mid-unit of S. For $a,\ b \in\ S$, we have $u^2ab = abu^2$ (by Proposition 5.1(1))
$= au^2b = ab$, as $u^2 \in M$ and S^2 thus possess the identity u^2.

In contrast, suppose that the subseminearring S^2 has a fixed identity, denoted by e. Considering any $x, y \in S$. Then

$$yex = e\,(yx)$$

$$= yx.$$

So because idempotent e is at the center of S. The proof is done since e is the mid-unit of S as a result.

Proposition 5.4. If u is a mid-unit of S, then $uS = (Su)$ also a duo-seminearring that possesses identity.

Proof. Suppose that u^2 is the identity of uS and $Su = S^2$. If R is any right ideal of S, then $RS = RuS \subseteq R$ and so R is an ideal of S. Specifically, R is an ideal of uS. The centrality of u, however, allows us to demonstrate, that any left ideal L of uS $(= Su)$, is consequently an ideal of uS. uS is a duo-seminearrings as a result.

Theorem 5.5. Assume S represents a mid-unit's duo-seminearring. Then, S/σ becomes a duo-seminearring with identity due to the existence of a separating congruence σ on S that is idempotent.

Proof. For $a, b \in S$, define $a\sigma b$ if $ua = ub$, for some $u \in M$. Clearly, if e, f are idempotents in S and σ is a congruence on S, then

$$\sigma fe \Longrightarrow ue = uf$$

$$\Longrightarrow e = eue = euue = fuuf = fuf = f^2 = f.$$

As a result, the idempotents of S are separated by σ. For each $u \in M,\ \sigma u = M$ and σu are the identity of S/σ. This proves that S/σ is a duo.

Corollary 5.6. An idempotent e of a duo-seminearring S is a mid-unit of S iff $eu = u^2$, for any $u \in M$.

Proof. This is trivial and directly follows from the Theorem 5.5.

6 Conclusion

The right duo-seminearring is specifically discussed in this study including the characterization of seminearring mid-units. We have also provided duo-seminearrings for mid-units. We have examined how idempotents and mid-units are related. Finally, the necessary and sufficient conditions are obtained to have a mid-units for a duo-seminearrings.

REFERENCES

[1] Van Hoorn, W. G., & Van Rootselaar, B. (1967). Fundamental notions in the theory of seminearrings. Compositio Mathematica, 18(1-2), 65-78.

[2] G. Pilz, Near-rings: The Theory and Its Applications, Revised edition, North Hollond, 1983.

[3] Yamada, M. (1955). A note on middle unitary semigroups. In Kodai Mathematical Seminar Reports (Vol. 7, No. 2, pp. 49-52). Department of Mathematics, Tokyo Institute of Technology.

[4] Ault, J. E. (1974). Semigroups with midunits. Transactions of the American Mathematical Society, 190, 375-384.

[5] Blyth, T. S. (1976). On middle units in orthodox semigroups. In Semigroup forum (Vol. 13, No. 1, pp. 261-265). Springer-Verlag.

[6] Manikandan, G. and Perumal, R. (2021). Mid-units in duo seminearrings. In Journal of Physics: Conference Series (Vol. 1850, No. 1, p. 012037). IOP Publishing.

[7] Balakrishnan, R., & Perumal, R. (2012). Left duo seminear-rings. Scientia Magna, 8(3), 115-120.

[8] Senthil, S., & Perumal, R. (2021, May). Minimal prime ideals in seminearrings. In Journal of Physics: Conference Series (Vol. 1850, No. 1, p. 012100). IOP Publishing.

[9] Senthil, S., Perumal, R., & Arulprakasam, R. (2020). On Structures of Right Duo Seminearring, Advances in Mathematics: Scientific Journal Vol. 9, no.11, 9231–9236.

[10] Hall, T.E., On regular semigroups whose idempotents form a subsemigroup, Bull. Australian Math. Soc., 1 (1969), 195-208.

[11] Koppula, K., Srinivas, K. B., & Prasad, K. S. (2020). On prime strong ideals of a seminearring. Mat. Vesnik, 72, 243-256.

[12] Krishna, K. V., & Chatterjee, N. (2018). A necessary condition to test the minimality of generalized linear sequential machines using the theory of near-semirings. Algebra and Discrete Mathematics, 4(3), 30-45.

[13] Krishna, K. V., & Chatterjee, N. (2007). Representation of near-semirings and approximation of their categories, Southeast Asian Bull. Math., 31 (2007), 903–914.

[14] Jagadeesha, B., Kuncham, S.P., & Kedukodi, B.S. (2016). Implications on a Lattice, Fuzzy. Inf. Eng., 8(4), 411–425.

[15] Nayak, H., Kuncham, S.P., & Kedukodi, B.S. (2019). Extensions of boolean rings and nearrings, Journal of Siberian Federal University – Mathematics and Physics, 12(1), 58–67.