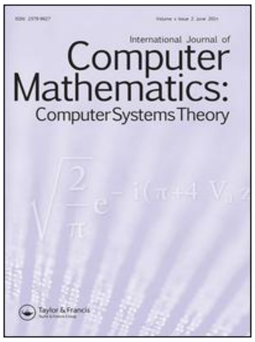

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Induced *H*-packing *k*-partition of graphs

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ABSTRACT

The minimum induced H-packing k-partition number is denoted by $ipp^{\mathscr{H}}(G,H)$. The induced H-packing k-partition number denoted by $ipp(G,H)$ is defined as $ipp(G,H) = \min ipp^{\mathscr{H}}(G,H)$ where the minimum is taken over all H-packings of G. In this paper, we obtain the induced P_3-packing k-partition number for trees, slim trees, split graphs, complete bipartite graphs, grids and circulant graphs. We also deal with networks having perfect $K_{1,3}$-packing where $K_{1,3}$ is a claw on four vertices. We prove that an induced $K_{1,3}$-packing k-partition problem is NP-Complete. Further we prove that the induced $K_{1,3}$-packing k-partition number of Q^r is 2 for all hypercube networks with perfect $K_{1,3}$-packing and prove that $ipp(LQ^r) = 4$ for all locally twisted cubes with perfect $K_{1,3}$-packing.

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1. Introduction

The *degree* or *valency* $d_G(v)$ of a vertex v in G is the number of edges of G incident with v, each loop counted as two edges. The *diameter* of G denoted by $d(G)$ is the maximum distance between two vertices of G. In other words, $d(G) = \max\{d_G(u,v) : u, v \in V(G)\}$. For each vertex $v \in V$, the *open neighbourhood* of v is the set $N(v)$ containing all the vertices u adjacent to v and the closed neighbourhood of v is the set $N(v) = N(v) \cup \{v\}$. A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. A *bipartite graph* is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y. Such a partition (X, Y) is called a bipartition of the graph. A graph is bipartite if and only if it contains no odd cycle. A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. The complete bipartite graph with partite sets of size m and n is denoted as $K_{m,n}$. An *H-packing* in a graph $G = (V, E)$ is a set of vertex disjoint induced subgraphs of G, called H-subgraphs, each of which is isomorphic to H. The vertices belonging to the H-subgraphs are said to be saturated by the H-subgraphs. The remaining vertices are unsaturated. A *perfect H-packing* in a graph G is a set of H-subgraphs of G such that every vertex in G is incident with one H-subgraph in this set. An almost perfect H-packing in a graph G is a set of H-subgraphs of G such that at most $|V(H)| - 1$ number of vertices are not incident on any H-subgraph in G [10]. The matching problem [2] is a particular case of packing problem when $H \simeq K_2$.

Partitioning a network with respect to vertices, edges or subgraphs is a significant aspect in enlarging resource utilization of parallel machines. Partitioning large networks is often important for complexity reduction or parallelization. For instance, in telecommunication networks, same frequency can be assigned to different subnetworks if the frequencies do not interfere with each other.

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Thus the study of partitioning a H-packing such that no two members in the same partition interfere becomes meaningful [10]. A collection $\mathscr{K} = \{H_1, H_2, \ldots, H_r\}$ of induced subgraphs of a graph G is said to be *sg-independent* if (i) $V(H_i) \bigcap V(H_j) = \phi$, $i \neq j$, $1 \leq i, j \leq r$ and (ii) no edge of G has its one end in H_i and the other end in H_j, $i \neq j$, $1 \leq i, j \leq r$. If $H_i \simeq H$, $\forall i$, $1 \leq i \leq r$, then $\mathscr{K}$ is referred to as a *H-independent set* of G. Let $\mathscr{H}$ be a perfect or almost perfect H-packing of a graph G. Finding a partition $\{\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_k\}$ of $\mathscr{H}$ such that $\mathscr{H}_i$ is H-independent set, $\forall i$, $1 \leq i \leq k$, with minimum k is called the induced H-packing k-partition problem of G. The minimum induced H-packing k-partition number is denoted by $ipp^{\mathscr{H}}(G, H)$. The induced H-packing k-partition number denoted by $ipp(G, H)$ is defined as $ipp(G, H) = \min ipp^{\mathscr{H}}(G, H)$ where the minimum is taken over all H-packing of G [10, 13, 14]. The induced H-packing k-partition problem was studied for certain interconnection networks such as hypercubes, Sierpiñski graphs [10]. Jesu Raja et al [10] proved that the induced P_3-packing k-partition problem is NP-complete, and the induced C_4-packing k-partition problem is NP-complete. An induced P_3-packing k-partition number was studied for butterfly networks, honeycomb networks and circum pyrene [14]. Xavier et al [14, 15] obtained the induced H-packing k-partition number for augmented cube, crossed cube, enhanced hypercube with $H \simeq P_3$ and C_4 and an H-packing and an induced H-packing k-partition number for V-phenylenic nanotube, H-naphtalenic nanotube, H-anthracenic nanotube, H-tetracenic nanotube, $CNC_3[n]$ nanocone and circum tetracene with $H \simeq P_3$. The objective and basic concept of the paper is to find H-packing and an induced H-packing k-partition number for certain graphs and networks. Hence we obtain the induced P_3-packing k-partition number for trees, slim trees, split graphs, complete bipartite graphs, grids and circulant graphs. We deal with networks having perfect $K_{1,3}$-packing where $K_{1,3}$ is a claw on four vertices. We prove that an induced $K_{1,3}$-packing k-partition problem is NP-Complete. Further we prove that the induced $K_{1,3}$-packing k-partition of Q^r is 2 for all hypercube networks with perfect $K_{1,3}$-packing and prove that $ipp(LQ^r) = 4$ for all locally twisted cubes with perfect $K_{1,3}$-packing.

Remark 1.1 ([10]): In the sequel, we represent the vertex set of $\mathscr{H}_i$ as V_i, $1 \leq i \leq k$ and the subgraph induced by V_i as $[V_i]$. Let $|[V_i]|$ denote the number of H-independent sets in $[V_i]$.

The following example illustrates the concept in cycle, when $H \simeq P_3$. A cycle is a closed path such that the start vertex and end vertex are the same. The cycle graph with n vertices is denoted by C_n. The number of vertices in C_n equals the number of edges, and every vertex has degree 2. A cycle is also referred to as ring architecture. A cycle is often used as a connection structure for local area networks, and can also be used as a control or data flow structure for distributed computation in arbitrary networks [3].

Example 1.1: Let C_n be a cycle of order n with perfect P_3-packing. Then

$$ipp(C_n, P_3) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

Proof: C_n has perfect P_3-packing, $n \equiv 0 \; mod \; 3$. Let $\mathcal{P} = \{H_1, H_2, \ldots, H_{n/3}\}$ be a perfect P_3-packing of C_n. This implies that each H_i is isomorphic to P_3, $1 \leq i \leq n/3$. Label the vertices of C_n in a clockwise direction with consecutive integers $1, 2, \ldots, n$.

Case i: n is even.

Then the paths $P^i : (6i-5, 6i-4, 6i-3)$, $1 \leq i \leq n/6$ are vertex disjoint P_3-paths in C_n. Clearly $[V_1] = \bigcup_{i=1}^{n/6} V(P^i)$ and $[V_2] = V \setminus [V_1]$ induce P_3- partition, where V denotes the vertex set of G. Hence $ipp(C_n, P_3) = 2$.

Case ii: n is odd.

The paths $P_1^i : (9i-8, 9i-7, 9i-6)$, $1 \leq i \leq n/9$ are vertex disjoint P_3-paths in C_n and the paths $P_2^i : (9i-5, 9i-4, 9i-3)$, $1 \leq i \leq n/9$ are also vertex disjoint P_3-paths in C_n. Clearly $[V_1] = \bigcup_{i=1}^{n/9}$

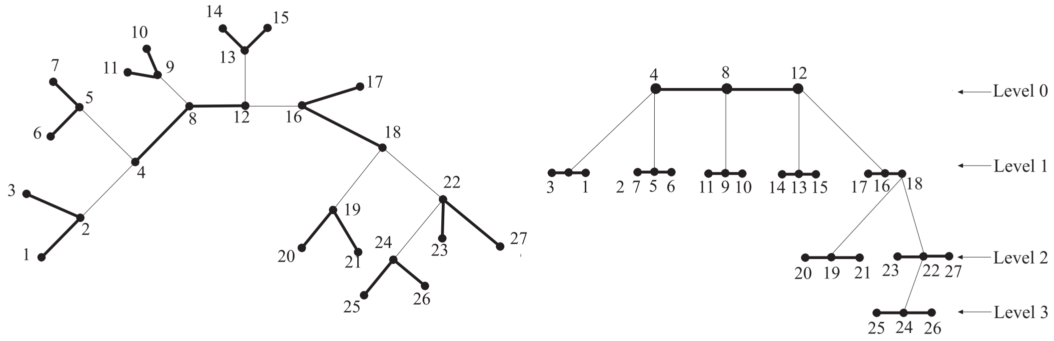

Figure 1. Tree T and Tree T^*.

$V(P_1^i)$, $[V_2] = \bigcup_{i=1}^{n/9} V(P_2^i)$ and $[V_3] = V \setminus [V_1] \cup [V_2]$ induce P_3- partition, where V denotes the vertex set of G. This implies $ipp(C_n, P_3) = 3$. ■

2. Induced P_3-packing k-partition of graphs

In this section we consider $H \simeq P_3$ and discuss the induced P_3-packing k-partition number for trees, slim trees, split graphs, complete bipartite graphs, grids and circulant graphs.

2.1. Trees

The network whose topological structure is a tree is called a tree network. A tree network is a common sight, simple, and easy to construct and expand. Its importance and interest are due to their simple structures and remarkable properties [8, 16]. We have the following theorem for trees with a perfect P_3-packing or an almost perfect P_3-packing.

Theorem 2.1 ([12]): *Let T be a tree with perfect P_3-packing. Then $ipp(T, P_3) = 2$.*

Proof: Let $\mathcal{P}$ be a perfect P_3-packing of T. For P^i, P^j in $\mathcal{P}$, $G[P^i \cup P^j]$, the graph induced by $P^i \cup P^j$ is a subtree of T on 5 edges or is the union of two disjoint subtrees P^i and P^j. We generate levels in $\mathcal{P}$ as follows:

Step 1: Arbitrarily select $P_0 \in \mathcal{P}$ at Level 0.

Step 2: Having selected members of $\mathcal{P}$ at Level i, for each such member P at Level i, include all members Q of $\mathcal{P}$, such that $|E(G[P \cup Q])| = 5$ at Level $i+1$. Call such members Q of $\mathcal{P}$ as children of P.

Step 3: Repeat Step 2 till all members in $\mathcal{P}$ are exhausted.

Replace each member of $\mathcal{P}$ in each of the levels by the corresponding P_3-paths. Add all the induced edges of T between levels i and $i+1$, $i \geq 0$. We claim that the tree T^* generated is isomorphic to T. There are no induced edges of T between members of $\mathcal{P}$ at the same level, as this would generate a cycle in T a contradiction. Again for the same reason there are no induced edges of T between children of two distinct members of $\mathcal{P}$ at the same level. This implies that $T^* \simeq T$. See Figure 1. Now partition $V(T)$ as $[V_1]$ and $[V_2]$, where $[V_1]$ contains all vertices at even levels of T^* and $[V_2]$ contains all vertices of odd levels of T^*. This implies $ipp(T, P_3) = 2$. ■

Remark 2.2: Let T be a tree with an almost perfect P_3-packing. Then $ipp(T, P_3) = 2$.

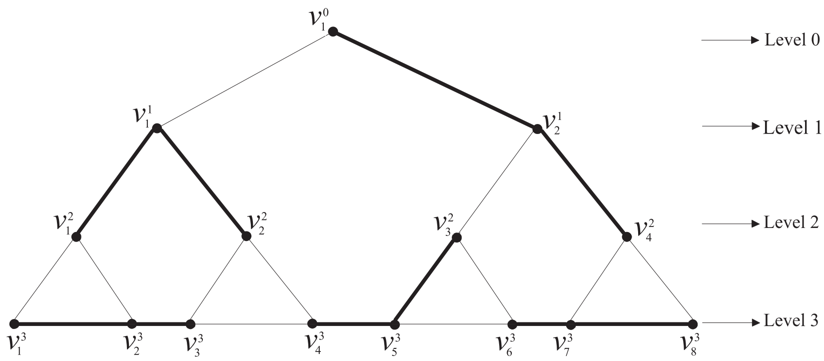

Figure 2. The slim tree $ST(3)$ with a P_3-packing.

2.2. Slim trees

Definition 2.3: The s^{th} slim tree $ST(s)$ is defined as a 5-tuple $ST(s) = (V, E, u, l, r)$, where V is the node set, E is the edge set, $u \in V$ is the root node, $l \in V$ is the left node, $r \in V$ is the right node, and s $i \geq 2$ is an integer. The s^{th} slim tree $ST(s)$ is recursively defined as follows:

(1) $ST(2)$ is the complete graph K_3 with its nodes labelled with u, l and r.
(2) The s^{th} slim tree $ST(s)$, with s $i \geq 3$, is composed of a root node u and two disjoint copies of $(s-1)$th slim trees as the left subtree and right subtree, denoted by $ST^l\ (s-1) = (V_1, E_1, u_1, l_1, r_1)$ and $ST^r\ (s-1) = (V_2, E_2, u_2, l_2, r_2)$, respectively, where in particular $u \notin V_1 \cup V_2$. To be specific, $ST(s) = (V, E, u, l, r)$ is given by $V = V_1 \cup V_2 \cup \{u\}$, $E = E_1 \cup E_2 \cup \{(u, u_1), (u, u_2), (r_1, l_2\}), l = l_1$, $r = r_2$.

By definition of $ST(s)$, the left subtree ST^l (s-1) and the right subtree ST^r (s-1) are isomorphic. This property is referred to as the symmetry property of $ST(s)$ [7].

Lemma 2.4: *The induced P_3-packing k-partition number of $ST(3)$ is 3, that is, $ipp(ST(3), P_3) = 3$.*

Proof: As $|V(ST(3))| = 15$, it is possible to pack $ST(3)$ with five vertex disjoint paths of length 2. One such packing is shown in Figure 2. Suppose $ipp(ST(3), P_3) = 2$. Let $[V_1]$ and $[V_2]$ be the induced P_3-packing 2-partition sets. Let path P: uvw be in $[V_1]$. Both u and w cannot be of degree 2. Without loss of generality, let $|N(u) \setminus \{v\}|$ $i \geq 1$ and $|N(w) \setminus \{v\}|$ $i \geq 2$. We have $(N(u) \setminus \{v\}) \cap (N(w) \setminus \{v\}) = \phi$. For otherwise the vertices u, v and w will induce a 3-cycle, a contradiction. Therefore, $|S| = |\ N(u) \cup N(v) \cup N(w)\ |$ $i \geq 3$. Suppose $|S| = 3$. These vertices are independent or induce an edge and an isolated vertex. Any path containing the vertices of the edge is in one partition set and isolated vertex in another partition set, a contradiction. This implies that $ipp(ST(3), P_3)$ $i \geq 3$. Now let $P = \{\ (v_1^0, v_2^1, v_4^2), (v_3^1, v_2^3, v_3^3\ )\}$, $Q = \{\ (v_1^2, v_1^1, v_2^2\ ), (v_6^3, v_7^3, v_8^3\ )\}$ and $R = \{\ (v_4^3, v_5^3, v_3^2\ )\}$. $P \cup Q \cup R$ is an optimal induced P_3-packing 3-partition in $ST(3)$. See Figure 2. Hence $ipp(ST(3), P_3) = 3$. ■

Theorem 2.5: *The induced P_3-packing k-partition number of $ST(s)$, $s > 2$ is 3, that is, $ipp(ST(s), P_3) = 3$.*

Proof: We prove the result by induction on the dimension s of slim tree $ST(s)$. We begin with $s = 4$. Let u be the root vertex of $ST(4)$. Now $ST(4) \setminus u$ comprises of two vertex disjoint copies of $ST(3)$. By Lemma 2.4, $ipp(ST(3), P_3) = 3$. Choosing u as the only unsaturated vertex in a packing, we get $ipp(ST(4)) = 3$. Removal of root vertex u of $ST(5)$ leaves two vertex disjoint copies of $ST(4)$. Partitioning the two vertex disjoint copies of $ST(4)$ as discussed above, leaves the root vertices of both the copies unsaturated. Further these root vertices can belong to at most two of the partition sets of $ST(5) \setminus u$. The root vertex u of $ST(5)$ together with the two root vertices of copies of $ST(4)$ in $ST(5)$, induce a P_3 which can be included in the third partition set of $ST(5) \setminus u$. Hence $ipp(ST(5)) = 3$. Assume the result to be true for $ST(s)$. Consider $ST(s+1)$. Suppose $s+1$ is even. $ST(s+1)$ contains two copies of $ST(s)$, say $ST(s)^1$ and $ST(s)^2$. By induction hypothesis $ipp(ST(s)^1, P_3) = ipp(ST(s)^2, P_3) = 3$, Select the partition sets of $ST(s)^2$ as that of $ST(s)^1$. This implies $ipp(ST(s+1), P_3) = 3$, leaving out one vertex unsaturated. Suppose $s+1$ is odd. Since s is even, by induction hypothesis $ST(s)$ has the root vertex as the only unsaturated vertex. The root vertex of $ST(s+1)$ together with the two root vertices of two copies of $ST(s)$ induce a P_3 which can be placed in at least one partition set. This implies $ipp(ST(s+1), P_3) = 3$. Therefore $ipp(ST(s), P_3) = 3$. ■

2.3. Split graphs

A graph $G = (V(G), E(G))$ is a split graph, if $V(G)$ can be partitioned into sets K and I, where K is a clique and I is an independent set [5].

Theorem 2.6: *Let G be the split graph with partition $V(G) = K + S$, where K is a complete graph and S is a stable set. Then the induced P_3-packing k-partition number lies between $\frac{|S|}{2} \leq ipp(G, P_3) \leq |S|$.*

Proof: There exists a perfect induced P_3-packing where $|K \cup S| \equiv 0 \bmod 3$. In K_m, there is no P_3 path which has both end vertices. This implies an induced P_3-paths has either one end in K and the other end in S or it has both ends in S. If all the induced P_3-paths saturate one vertex of S then the induced P_3-packing k-partition number of G is $|S|$. See Figure 3(b). On other hand, if all induced P_3-paths has both end in S, the induced P_3-packing k-partition number of G is $\frac{|S|}{2}$. See Figure 3(a). This implies $\frac{|S|}{2} \leq ipp(G, P_3) \leq |S|$. ■

2.4. Grid networks

Grid network topology is one of the key network architectures in which devices are connected with many redundant interconnections between network nodes such as routers and switches. A two-dimensional rectangular grid graph is the cartesian product of path graphs $P_m \times P_n$ where P_k is a

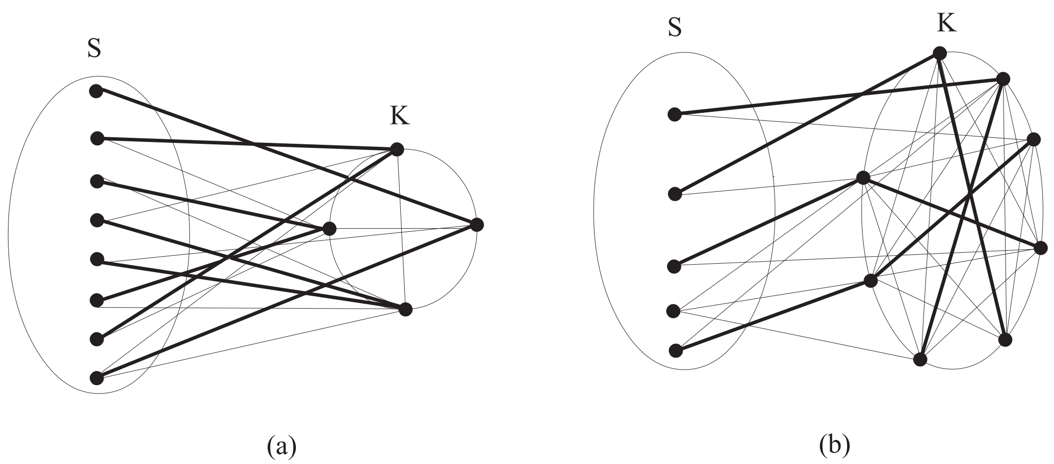

Figure 3. (a) Bold edges shows that the induced P_3-packing k-partition number of G is $\frac{|S|}{2}$ (b) Bold edges shows that the induced P_3-packing k-partition number of G is $|S|$.

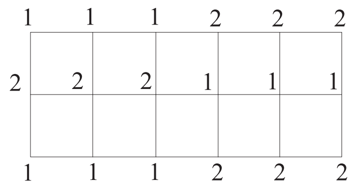

Figure 4. An induced P_3-packing 3-partition of $G_{m\times n}$.

path on k vertices and is denoted by $G_{m\times n}$. A vertex in the i^{th} row and j^{th} column of $G_{m\times n}$ is labelled as (i, j), $1 \leq i \leq m$, $1 \leq j \leq n$. It is observed that the square grid has diameter $2n{-}2$ [16].

Lemma 2.7: *Let $G_{m\times n}$ be a grid, where m and n are multiples of 3, $m, n \geq 3$. Then $G_{m\times n}$ has a perfect P_3-packing.*

Proof: Let $n = 3k$, $k \geq 1$. Then for $1 \leq i \leq m$ { $(i, 3j-2)$, $(i, 3j-1)$, $(i, 3j)$ }, $1 \leq j \leq n$ defines a P_3-packing of $G_{m\times n}$. ■

The following algorithm proves that $ipp(G_{m\times n}) = 2$

Procedure A: INDUCED P_3-PACKING 2-PARTITION ($G_{m\times n}$, $m, n \geq 3$)

Input: A grid of order $m \times n$, where m and n are multiple of 3.

Algorithm:

Label the vertex (i, j) as

(i) 1 if (j *mod* 6 $\in${1,2,3} and $i \equiv 1$ *mod* 2) or (j *mod*6 $\notin \{1, 2, 3\}$ and $i \equiv 0$ *mod* 2).

(ii) 2 if (j *mod*6 $\notin \{1, 2, 3\}$ and $i \equiv 1$ *mod* 2) or (j *mod* 6 $\in${1,2,3} and $i \equiv 0$ *mod* 2). See Figure 4.

End INDUCED P_3-PACKING 2-PARTITION

Output: $ipp(G, P_3) = 2$

Proof of correctness: Let V be the vertex set of G. Let $[V_1]$ contain the vertices $(i, 3j-2)$, $(i, 3j-1)$, $(i, 3j)$ whenever $i \equiv 1$ *mod* 2

$$j = \begin{cases} 1, 3, \ldots, \dfrac{n}{3} & \text{if } n \text{ is a odd multiples of 3.} \\ 1, 3, \ldots, \dfrac{n}{3} - 1 & \text{if } n \text{ is a even multiples of 3.} \end{cases}$$

and let $[V_2] = V \setminus [V_1]$. In other words the labelling induces in each row a sequence of three consecutive vertices labelled as 1 and the next three consecutive vertices as two alternatively beginning with 111 in odd rows and beginning with 222 in even rows. Let $[V_1]$ be the set of all vertices labelled as 1 and let $[V_2]$ be the set of all vertices labelled as 2. Clearly $[V_1]$ and $[V_2]$ are an induced P_3-packing 2-partition of $G_{m\times n}$ and hence $ipp(\, G_{m\times n}, P_3) = 2$ whenever m and n are multiples of 3.

Procedure B: INDUCED P_3-PACKING k-PARTITION ($G_{m\times n-1}$ and $G_{m-1\times n}$ $m, n \geq 3$, m and n are multiples of 3).

Input: A grid of order $G_{m\times n-1}$, $m, n \geq 3$.

Algorithm:

Label the vertices of $G_{m\times n-3}$ be labelled as

(i) 1 if (j *mod* 6 $\in${1,2,3}) and $i \equiv 1$ *mod* 2) or (j *mod*6 $\notin \{1, 2, 3\}$ and $i \equiv 0$ *mod* 2) and (i *mod*6 $\notin$ $\{1, 2, 3\}$ and $j \equiv 1$ *mod* 2)

(ii) 2 if ($j\ mod\ 6 \in \{1,2,3\}$ and $i \equiv 0\ mod\ 2$).

(iii) 3 if ($j\ mod6 \notin \{1,2,3\}$ and $i \equiv 1\ mod\ 2$) and ($i\ mod6 \notin \{1,2,3\}$ and $i \equiv 0\ mod\ 2$).

Label the $(n-2)$ column using the sequence 333,111,333$\cdots$,111 if m is even multiples of 3 and using the sequence 333,111,333$\cdots$,111 if m is odd multiples of 3. Label the $(n-1)$ columns as in $(n-2)$ by replacing 1 by 3 and 3 by 1.
End INDUCED P_3-PACKING k-PARTITION
Output: $ipp(G) \leq 3$
Proof of correctness is similar to that of procedure A. Therefore $ipp(G_{m\times n}, P_3) \leq 3$.

2.5. Circulant graphs

A circulant undirected graph, denoted by $G(n; \hat{A} \pm S)$ where $S \subseteq \{1, 2, \ldots, n/2\}, n \geq 3$ is defined as a graph consisting of the vertex set $V = \{0, 1, \ldots, n-1\}$ and the edge set $E = (i, j) : |j - i| = s(modn)$, $s \in S$. It is clear that $G(n; \hat{A} \pm 1)$ is the undirected cycle C_n and $G(n; \hat{A} \pm 1, 2, \ldots, \lfloor n/2 \rfloor)$ is the complete graph K_n. Further $G(n; \hat{A} \pm 1, 2, \ldots, j)$, $1 \leq j < n/2$, $n \geq 3$ is a $2j$ -regular graph [16].

Theorem 2.8: *Let G be the circulant graph $G(n, S)$ where $S = \{1, 2\}$. Then*

$$ipp(G, P_3) = \begin{cases} 3 & \text{If } n \equiv 3(mod6) \\ 2 & \text{otherwise} \end{cases}$$

Proof: Let $n = 6k$ or $6k + 3$. $G(n \pm \{1, 2\})$ is comprised of the outer cycle of length n and two disjoint inner cycle of length $n/2$. We define subsets of vertices $[V_1]$, $[V_2]$ and $[V_3]$ as follows: Without loss of generality, let $0, 1, 2 \in [V_1]$. Then either $3, 4, 5 \in [V_2]$ or $3, 4, 6 \in [V_2]$ or $3, 5, 7 \in [V_2]$. If $3, 4, 6 \in [V_2]$ then 5 must be in $[V_3]$ or if $3, 6, 9 \in [V_2]$ then 4 must be in $[V_3]$. On the other hand if $3, 4, 5 \in [V_2]$, labelling in the clockwise direction. We get $[V_1] = \{0, 1, 2, 6, 7, 8, \ldots, n-5, n-4, n-3\}$, $[V_2] = \{3, 4, 5, 9, 10, 11, \ldots, n-2, n-1, n\}$, if $n = 6k$. Similarly $[V_1] = \{0, 1, 2, 6, 7, 8, \ldots, n-8, n-7, n-6\}$, $[V_2] = \{3, 4, 5, 9, 10, 11, \ldots, n-5, n-4, n-3\}$ and $[V_3] = \{n-2, n-1, n\}$, if $n = 6k + 3$. ∎

$G(n \pm \{1, 2\})$ when $n = 6k + 1, 6k + 2, 6k + 3$ is a near perfect graph. Since G is vertex transitive, it is enough to consider $G \setminus \{u, v\}$ for any $u, v \in V$.

Theorem 2.9: *For any positive integer m, there exists a connected graph G such that $ipp(G, P_3) = m$.*

Proof: Let K_m be the complete graph on m vertices. With every vertex of K_m, identify a pendant vertex of a path of length 2 to obtain a new graph G on $3m$ vertices, where $V(G) = V(G) \cup (\cup_{i=1}^{m} \{ a_i', a_i'', \})$ and $E(G) = E(G) \cup (\cup_{i=1}^{m} \{a_i, a_i'), (a_i'\ a_i'')\}$. See Figure 5. We claim that $ipp(G, P_3) = m$. The collection of all induced P_3-paths gives a perfect P_3-packing of G. Since non pendant vertex of every induced P_3-path is adjacent to the non pendant vertex of any other P_3-path, the induced P_3-packing partition number of G is m. ∎

Theorem 2.10: *Let $K_{m,n}$ be the complete bipartite graph with a perfect or almost perfect induced P_3-packing. Then the induced P_3-packing k-partition number is $\lfloor \frac{m+n}{3} \rfloor$.*

Proof: Let X and Y be the bipartite sets of $K_{m,n}$ with $|X| = m$ and $|X| = n$. No two vertex disjoint P_3-paths in $K_{m,n}$ can be in the same partition set, as there are at least three edges between them. Therefore the induced P_3-packing partition number of $K_{m,n}$ is $\lfloor \frac{m+n}{3} \rfloor$. See Figure 6. ∎

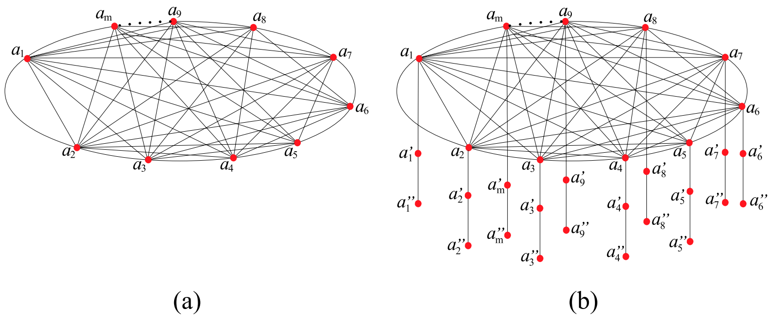

Figure 5. (a) Complete graph K_m (b) G defined in Theorem 2.9.

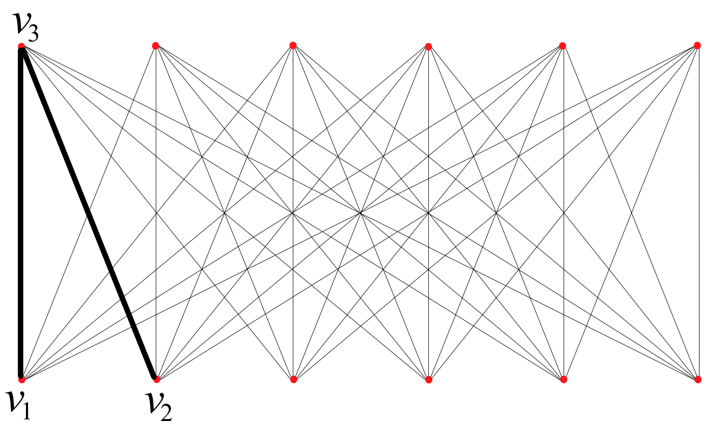

Figure 6. Complete bipartite graph $K_{6,6}$.

3. Induced $K_{1,3}$-packing k-partition problem

One of the most widely studied packing is claw-packing. A claw is another name for the complete bipartite graph $K_{1,3}$. A claw-free graph is a graph in which no induced subgraph is a claw [4].

3.1. NP-completeness

We shall show that induced $K_{1,3}$-packing k-partition problem (*ipp*) is in the class *NP*. To prove that the problem is *NP*-complete, we exhibit a polynomial reduction from the chromatic index problem which is known to be an *NP*-complete problem. The chromatic index $\chi'(G)$ is the least number of colours required to colour the vertices of G properly in such a way that no two adjacent vertices have the same colour. It is enough to set up a one to one correspondence between an already known *NP*- complete problem and induced $K_{1,3}$-packing k-partition problem. We have chosen to build a reduction from chromatic index problem. Suppose $G = (V, E)$ is an arbitrary instance of chromatic index problem. We must construct a graph $G^* = (V^*, E^*)$ with an induced $K_{1,3}$-packing k-partition if and only if $G = (V, E)$ has a chromatic index k.

Theorem 3.1: *The INDUCED $K_{1,3}$-PACKING k-PARTITION problem is NP-Complete.*

Proof: Let $G = (V, E)$ be the graph with n vertices. G^* is obtained from G by identifying every vertex of G with the 3-degree vertex of a $K_{1,3}$. The resulting graph $G^* = (V^*, E^*)$ has $4n$ vertices.

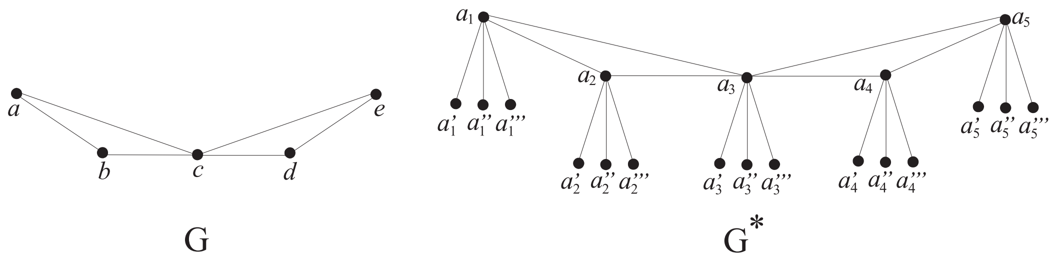

Figure 7. Gadget for chromatic index into induced $K_{1,3}$-packing k-partition.

See Figure 7. Thus $G^* = (V^*, E^*)$ has $V(G^*) = V(G) \cup (\cup_{i=1}^{n} \{ a_i', a_i'', a_i''' \})$ and $E(G^*) = E(G) \cup \{(a_i, a_i'), (a_i\ a_i''), (a_i, a_i''')/1 \le i \le n\}$.

Clearly every vertex in G is identified with an induced claw on four vertices in G^*. We claim that G has chromatic index λ if and only if $ipp(G^*, K_{1,3}) = \lambda$. Let the chromatic index of G be λ. Let $V_1, V_2, \ldots, V_\lambda$ be the colour sets of G. Let V_i^* denote all copies of $K_{1,3}$-packing's where vertices of degree 3 are coloured i, $1 \le i \le \lambda$. Then $V_1^*, V_2^*, \ldots, V_\lambda^*$ is a $K_{1,3}$-packing λ-partition of G^*. Conversely delete the pendant edges of every $K_{1,3}$ in V_i^* and colour the root of the corresponding $K_{1,3}$ as i, $1 \le i \le \lambda$. The resulting graph in G has chromatic index λ. This implies that the induced $K_{1,3}$-packing k-partition problem is NP-Complete. ■

Theorem 3.2: *For every positive integer $m \ge 4$, there exists a graph G with $ipp(G, K_{1,3}) = m$.*

Proof: Let K_m be the complete graph on m vertices. With every vertices of K_m, identifying a $K_{1,3}$ to obtain a new graph G on $4m$ vertices, where $V(G) = V(G) \cup (\cup_{i=1}^{n} \{ a_i', a_i'', a_i''' \})$ and $E(G) = E(G) \cup (\{ (a_i, a_i'), (a_i\ a_i''), (a_i, a_i''') \}, 1 \le i \le m$. See Figure 8. We claim that $ipp(G, K_{1,3}) = m$. The collection of all induced $K_{1,3}$ gives a perfect $K_{1,3}$-packing of G. Vertices of degree 3 in each copy of $K_{1,3}$ in the packing induce a complete graph in G. This implies that the induced $K_{1,3}$-packing partition number of G is m. ■

Theorem 3.3: *Let $K_{m,n}$ be the complete bipartite graph with a perfect or almost perfect induced $K_{1,3}$-packing. Then the induced $K_{1,3}$-packing k-partition number is $\lfloor \frac{m+n}{4} \rfloor$.*

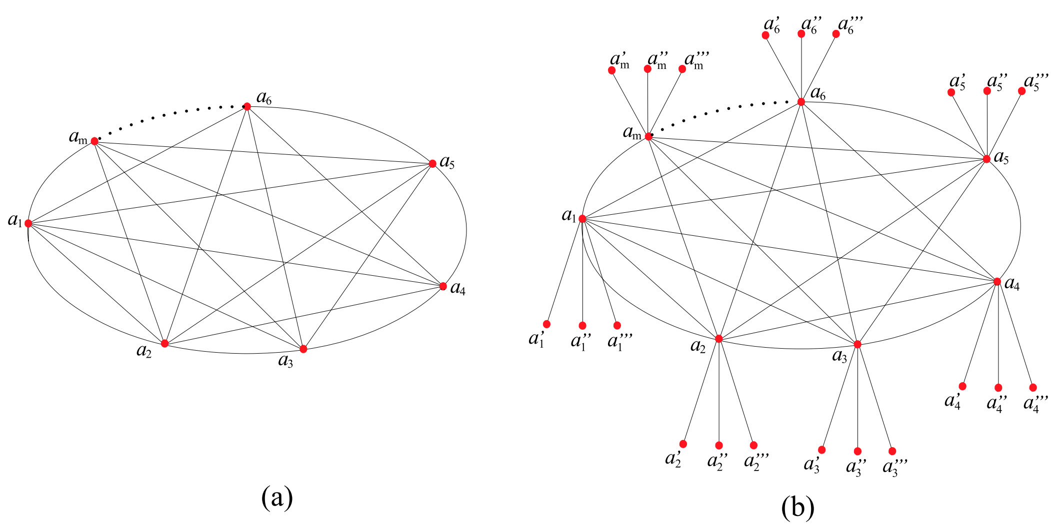

Figure 8. (a) Complete graph K_m (b) G defined in Theorem 3.2.

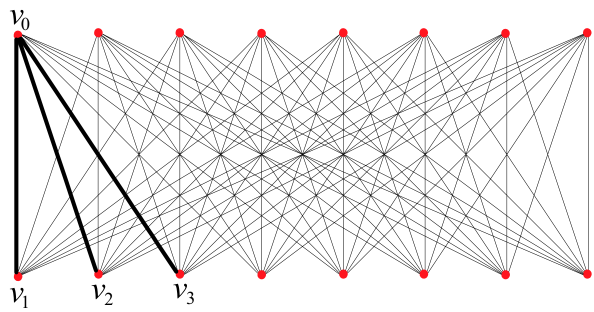

Figure 9. Complete bipartite graph $K_{8,8}$.

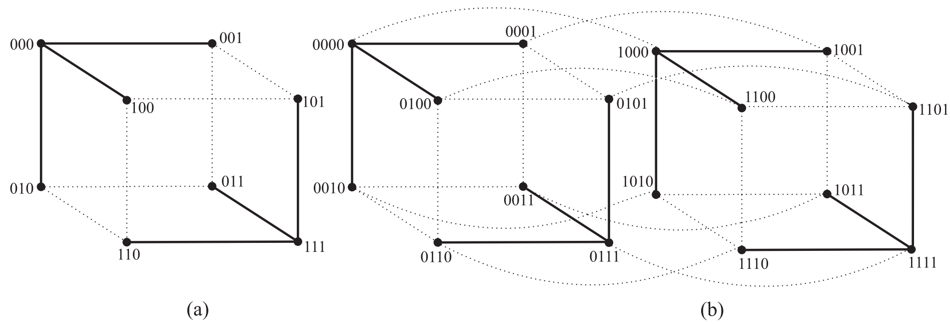

Figure 10. (a) An induced $K_{1,3}$-packing 2-partition number of Q^3 (b) An induced $K_{1,3}$-packing 3-partition number of Q^4.

Proof: Let X and Y be the bipartite sets of $K_{m,n}$ with $|X| = m$ and $|X| = n$. No two vertex disjoint $K_{1,3}$-claw packing in $K_{m,n}$ can be in the same partition set, as there are at least 4 edges between them. Therefore the induced $K_{1,3}$-packing partition number of $K_{m,n}$ is $\lfloor \frac{m+n}{4} \rfloor$. See Figure 9. ∎

3.2. Hypercube networks

For $n \geq 1$, let Q^n denote an n-dimensional binary cube where the nodes of Q^n are all the binary n-tuples and two nodes are adjacent if and only if their corresponding n-tuples differ in exactly one position. Two vertices $x, y \in V(Q^n)$ are adjacent if and only if the corresponding vectors differ exactly in one entry. For convenience, the labels $\{0, 1, 2, \ldots, 2^n - 1\}$ of Q^n are represented by $\{1, 2, \ldots, 2^n\}$, respectively [1, 6, 9, 10]. See Figure 10.

3.3. $K_{1,3}$-packing of hypercube networks

Theorem 3.4: *For r i ≥ 3, let Q^r be the hypercube network. Then Q^r has a perfect $K_{1,3}$-packing.*

Proof: The proof of the result uses the same technique as in [10]. We prove the result by induction on the dimension r of the hypercube network Q^r. We begin with $r = 3$. $\mathscr{P}^3 = \{ (000, 010, 100, 001), (111, 110, 011, 101) \}$ is a perfect $K_{1,3}$-packing. In Q^4, $\mathscr{P}^4 = 0\mathscr{P}^3 \cup 1\mathscr{P}^3$ is a perfect $K_{1,3}$-packing where $i\ \mathscr{P}^3$ denotes the set of $K_{1,3}$ in $\mathscr{P}^3$ prefixed by i, $i = 0,1$. See Figure 10(b). Assume the result to be true for Q^r. Consider Q^{r+1}. By induction hypothesis each copy of Q^r in Q^{r+1} contains a perfect $K_{1,3}$-packing. The union is a perfect $K_{1,3}$-packing in Q^{r+1}, that is $\mathscr{P}^{r+1} = 0\mathscr{P}^r \cup 1\mathscr{P}^r$. ∎

3.4. Induced $K_{1,3}$-packing k-partition number of hypercube networks

Lemma 3.5: *The induced $K_{1,3}$-packing k-partition number of Q^3 is 2, that is, $ipp(Q^3, K_{1,3}) = 2$.*

Proof: The proof of the result uses the same technique as in [10]. Without loss of generality let $H_1 \simeq K_{1,3}$, H_1:v_1 v_2 v_3 v_4 is a claw on four vertices in Q^3, where $d_H(v_1) = 3$, $d_H(v_2) = d_H(v_3) = d_H(v_4) = 1$. Then $|\bigcup_{i=1}^{4} N(v_i)| = 4$. Hence another claw H_2 on four vertices such that $V(H_1) \cap V(H_2) = \emptyset$ contains at least one vertex from $|\bigcup_{i=1}^{4} N(v_i)|$. This implies that $ipp(Q^3)$ $i \geq 2$. Now let $H_1 = \{ (000, 010, 100, 001) \}$ and $H_2 = \{ (111, 110, 011, 101) \}$. $H_1 \cup H_2$ is an optimal induced $K_{1,3}$-packing 2-partition in Q^3. ■

Lemma 3.6: *The induced $K_{1,3}$-packing k-partition number of Q^r is 2, that is, $ipp(Q^r, K_{1,3}) = 2$.*

Proof: We prove this results by induction on the dimension r of the hypercube network Q^r. We begin with $r = 5$. Q^5 contains four copies of Q^3, say Q_1^3, Q_2^3, Q_3^3, Q_4^3. Let $[V_1^i]$, $[V_2^i]$ be the induced $K_{1,3}$-packing 2-partition sets of $Q_i^3, 1 \leq i \leq 4$. We now claim that the binding edges in $(Q_1^3 \cup Q_2^3) \setminus Q_1^3$ incident at vertices of $[V_i^1], 1 \leq i \leq 2$, have their other ends in exactly one $[V_j^2], 1 \leq j \leq 2$. Suppose not, without loss of generality let all the end vertices of binding edges incident at vertices of $[V_1^1]$ be adjacent to vertices in $[V_1^2]$ and $[V_2^2]$. Then no vertex in $[V_2^1]$ is adjacent to any vertex in $[V_1^2]$, a contradiction. This argument is also true for $[V_i^1]$, $i = 2$. This implies that the binding edges incident at vertices of $[V_i^1], 1 \leq i \leq 2$, have their other ends in exactly one $[V_j^2], 1 \leq j \leq 2$ in Q^5. By Lemma 3.5, $ipp(Q^3, K_{1,3})$ is 2. Let $[V_1]$, $[V_2]$ be the induced $K_{1,3}$-packing 2-partition sets of Q_1^3. Without loss of generality, let each of $[V_1^1]$, $[V_2^1]$ contain at most four vertices of $V(Q_1^3)$. Let $[V_1] = \{ H_1 \}$, where H_1 is the graph induced by u_1,u_2,u_3 and u_4 with $d_{H_1}(u_1) = 3$, $d_{H_1}(u_2) = d_{H_1}(u_3) = d_{H_1}(u_4) = 1$ in Q_1^3. Then $|\bigcup_{i=1}^{4} N(u_i) \cap Q_2^3| = 4$. Hence $\bigcup_{i=1}^{4} N(u_i) \cap Q_2^3$ is not in $[V_1]$. This implies $\bigcup_{i=1}^{4} N(u_i) \cap Q_2^3$ is in $[V_2]$. Let $[V_2] = \{ H_2 \}$, where H_2 is the graph induced by v_1,v_2,v_3 and v_4 with $d_{H_2}(v_1) = 3$, $d_{H_2}(v_2) = d_{H_2}(v_3) = d_{H_2}(v_4) = 1$ in Q_1^3. Then $|\bigcup_{i=1}^{4} N(v_i) \cap Q_2^3| = 4$. Hence $\bigcup_{i=1}^{4} N(v_i) \cap Q_2^3$ is not in $[V_2]$. This implies $\bigcup_{i=1}^{4} N(v_i) \cap Q_2^3$ is in $[V_1]$. Similarly Q_3^3 is partitioned as in Q_2^3 and Q_4^3 is partitioned as in Q_1^3. Now $[V_1^i] \cup [V_2^i]$, $i = 1, 2, 3, 4$ is an optimal induced $K_{1,3}$-packing 2-partition in Q^5. Assume that the result is true for Q^{r-1}. Q^r contains two copies of Q^{r-1}, say Q_1^{r-1} and Q_2^{r-1}. By the induction hypothesis $ipp(Q_1^{r-1})$ is 2. Since $Q_1^{r-1} \simeq Q_2^{r-1}$, $[V_1^i] \cup [V_2^i]$, $i = 1, 2$ is an optimal induced $K_{1,3}$-packing 2-partition in Q^r. Then $ipp(Q^r, K_{1,3}) = 2$. See Figure 11. ■

3.5. Locally twisted cubes

A suitable interconnection network is an important part for the design of a multicomputer or multiprocessor system. This network is usually modelled by a symmetric graph, where the nodes represent the processing elements and the edges represent the communication channels. Desirable properties of an interconnection network include symmetry, embedding capabilities, relatively small degree, small diameter, scalability, robustness, and efficient routing. The locally twisted cubes is a better hypercube variant which is conceptually closer to hypercube than existing variants. One advantage is that the diameter of locally twisted cube is only about half the diameter of hypercubes [17]. The n-dimensional locally twisted cube LTQ^n ($n \geq 2$) is defined recursively as follows.

(a) LTQ^2 is a graph isomorphic to Q^2.
(b) For $n \geq 3$, LTQ^n is built from two disjoint copies of LTQ^{n-1} according to the following steps. Let $0LTQ^{n-1}$ denote the graph obtained by prefixing the label of each vertex of one copy of LTQ^{n-1} with 0, let $1LTQ^{n-1}$ denote the graph obtained by prefixing the label of each vertex of the other copy LTQ^{n-1} with 1, and connect each vertex $x = 0x_2x_3 \cdots x_n$ of $0LTQ^{n-1}$ with the vertex $1(x_2 + x_n)x_3 \cdots x_n$ of $1LTQ^{n-1}$ by an edge, where $+$ represents the modulo 2 addition. The graphs shown in Figure 12(a) and (b) are LTQ^3 and LTQ^4, respectively [11, 17].

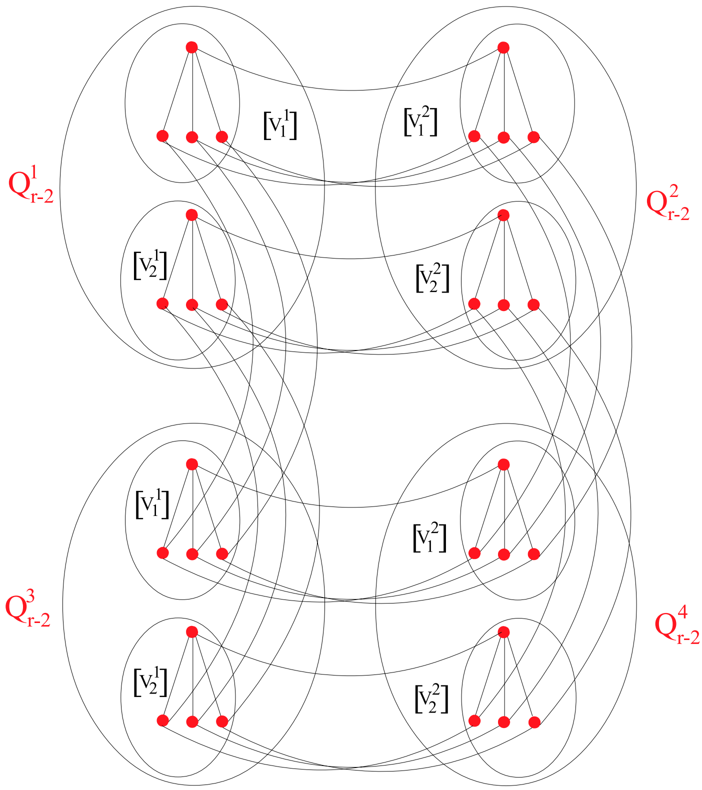

Figure 11. Possibilities of binding edges in Q^r.

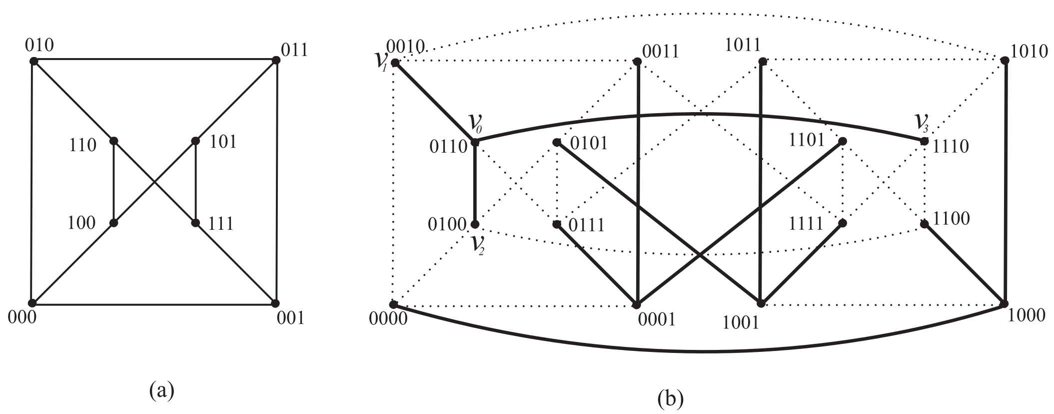

Figure 12. (a) Locally twisted cubes LTQ^3 (b) An induced $K_{1,3}$-packing 4-partition number of LTQ^4.

3.6. Induced $K_{1,3}$-packing k-partition of locally twisted cubes

Theorem 3.7: *The locally twisted cube LTQ^n of dimension n has a perfect $K_{1,3}$-packing for n i ≥ 4.*

Proof: The proof of the result uses the same technique as in [10]. We prove the result by induction on the dimension n of the hypercube network LTQ^n. We begin with $n = 4$. $\mathscr{P}^4$={(1000, 0000, 1010, 1100), (0110, 0010, 0100, 1110), (1001, 0101, 1011, 1111), (0001, 0111, 0011, 1101)}is a perfect $K_{1,3}$-packing. In LTQ^5, $\mathscr{P}^5 = 0\mathscr{P}^4 \cup 1\mathscr{P}^4$ is a perfect $K_{1,3}$-packing where i $\mathscr{P}^4$

denotes the set of $K_{1,3}$ in $\mathscr{P}^4$ prefixed by i, $i = 0, 1$. See Figure 12(b). Assume the result to be true for LTQ^n. Consider LTQ^{n+1}. By induction hypothesis each copy of LTQ^n in LTQ^{n+1} contains a perfect $K_{1,3}$-packing. The union is a perfect $K_{1,3}$-packing in LTQ^{n+1}, that is $\mathscr{P}^{n+1} = 0\mathscr{P}^n \cup 1\mathscr{P}^n$. ■

Lemma 3.8: *The induced $K_{1,3}$-packing k-partition number of LTQ^4 is 4, that is, $ipp(LTQ^4, K_{1,3}) = 4$.*

Proof: LTQ^4 is packed with 4 vertex disjoint claws on four vertices. Let H be a $K_{1,3}$ on vertices v_0, v_1, v_2, v_3 with $d_H\ (v_0) = 3$. See Figure 12(b). Let $S = \{\ v_0, v_1, v_2, v_3\ \}$. It is easy to verify that that $|N(S)| = 7$. In other words $|N[S]| = 11$. Hence any other copy of $K_{1,3}$ in LTQ^4 shares a vertex with $|N[S]|$. Hence no two copies of $K_{1,3}$ lie in the same partition set. Therefore $ipp(LTQ^4, K_{1,3}) = 4$. ■

Theorem 3.9: *The induced $K_{1,3}$-packing k-partition number of LTQ^r satisfies $ipp(LTQ^r, K_{1,3}) = 4$.*

Proof: We prove the result by induction on the dimension r of the locally twisted cube LTQ^r. Let r be odd. We begin with $r = 5$. LTQ^5 contains two copies of LTQ^4, say LTQ_1^4, LTQ_2^4. $LTQ_1^4 \cup LTQ_2^4$ contains four copies of Q^3, say Q_1^3, Q_2^3, Q_3^3, Q_4^3, that is $LTQ_1^4 = Q_1^3 \cup Q_2^3$ and $LTQ_2^4 = Q_3^3 \cup Q_4^3$. By Lemma 3.8, $ipp(LTQ^4, K_{1,3})$ is 4. Let $[V_1]$, $[V_2]$, $[V_3]$, $[V_4]$ be the induced $K_{1,3}$-packing 4-partition sets of LTQ_1^4. By Lemma 3.6, let $[V_1]$, $[V_2]$ be the induced $K_{1,3}$-packing 2-partition sets of Q_1^3 and let $[V_3]$, $[V_4]$ be the induced $K_{1,3}$-packing 2-partition sets of Q_2^3 in LTQ_1^4. Similarly Q_3^3 is partitioned by $[V_3]$, $[V_4]$ and Q_4^3 is partitioned by $[V_1]$, $[V_2]$ in LTQ_2^4. Since $Q_1^4 \simeq Q_4^4$ and $Q_2^4 \simeq Q_3^4$, $ipp(LTQ^5, K_{1,3})\ i \geq 4$. let r be even. LTQ^6 contains four copies of LTQ^4, say LTQ_1^4, LTQ_2^4, LTQ_3^4, LTQ_4^4. By Lemma 3.8, $V(LTQ_1^4)$ is partitioned into $[V_1]$, $[V_2]$, $[V_3]$, $[V_4]$ such that each of $[V_i], 1 \leq i \leq 4$ contains 4 vertices of LTQ_1^4. We have $|[V_i]| = 1, 1 \leq i \leq 4$.

Consider the subgraph induced by $V(LTQ_1^4) \bigcup V(LTQ_2^4)$. Let $[V_1] = \{\ H_1\ \}$, where H_1 is the graph induced by u_1,u_2,u_3 and u_4 with $d_{H_1}(u_1) = 3$, $d_{H_1}(u_2) = d_{H_1}(u_3) = d_{H_1}(u_4) = 1$ in LTQ_1^4. Then $|\bigcup_{i=1}^4\ N(u_i)\ \cap LTQ_2^4| = 4$. Hence $\bigcup_{i=1}^4\ N(u_i)\ \cap LTQ_2^4$ is not in $[V_1]$. This implies $\bigcup_{i=1}^4$ $N(u_i)\ \cap LTQ_2^4$ in $[V_2]$. Let $[V_2] = \{\ H_2\ \}$, where H_2 is the graph induced by v_1,v_2,v_3 and v_4 with $d_{H_2}(v_1) = 3$, $d_{H_2}(v_2) = d_{H_2}(v_3) = d_{H_2}(v_4) = 1$ in LTQ_1^4. Then $|\bigcup_{i=1}^4\ N(v_i)\ \cap LTQ_2^4| = 4$. Hence $\bigcup_{i=1}^4 N(v_i)\ \cap LTQ_2^4$ is not in $[V_2]$. This implies $\bigcup_{i=1}^4\ N(v_i)\ \cap LTQ_2^4$ in $[V_1]$. Let $[V_3] = \{w_1,w_2,w_3,w_4\}$ $[V_3] = \{x_1,x_2,x_3,x_4\}$. Since $|\bigcup_{i=1}^4\ N(w_i)\ \cap LTQ_2^4| = 0$ and $|\bigcup_{i=1}^4\ N(x_i)\ \cap LTQ_2^4| = 0$ the role of the partition sets $[V_3]$ and $[V_4]$ in LTQ_1^4 is the same as that of $[V_3]$ and $[V_4]$ in LTQ_2^4.

Now consider the subgraph induced by $V(LTQ_1^4) \bigcup V(LTQ_3^4)$. Let $[V_1] = \{\ H_1\ \}$, where H_1 is the graph induced by u_1,u_2,u_3 and u_4 with $d_{H_1}(u_1) = 3$, $d_{H_1}(u_2) = d_{H_1}(u_3) = d_{H_1}(u_4) = 1$ in LTQ_1^4. Then $|\bigcup_{i=1}^4\ N(u_i)\ \cap LTQ_3^4| = 4$. Hence $\bigcup_{i=1}^4\ N(u_i)\ \cap LTQ_3^4$ is not in $[V_1]$. Without loss of generality, let $\bigcup_{i=1}^4\ N(u_i)\ \cap\ LTQ_3^4$ in $[V_3]$. Let $[V_2] = \{\ H_2\ \}$, where H_2 is the graph induced by v_1,v_2,v_3 and v_4 with $d_{H_2}(v_1) = 3$, $d_{H_2}(v_2) = d_{H_2}(v_3) = d_{H_2}(v_4) = 1$ in LTQ_1^4. Then $|\bigcup_{i=1}^4\ N(v_i)\ \cap LTQ_3^4| = 4$. Hence $\bigcup_{i=1}^4\ N(v_i)\ \cap LTQ_3^4$ is not in $[V_2]$. Without loss of generality, let $\bigcup_{i=1}^4\ N(v_i)\ \cap LTQ_3^4$ in $[V_4]$. Let $[V_3] = \{\ H_3\ \}$, where H_3 is the graph induced by w_1,w_2,w_3 and w_4 with $d_{H_3}(w_1) = 3$, $d_{H_3}(w_2) = d_{H_3}(w_3) = d_{H_3}(w_4) = 1$ in LTQ_1^4. Then $|\bigcup_{i=1}^4\ N(w_i)\ \cap LTQ_3^4| = 4$. Hence $\bigcup_{i=1}^4\ N(w_i)\ \cap LTQ_3^4$ is not in $[V_3]$. This implies $\bigcup_{i=1}^4\ N(w_i)\ \cap\ LTQ_3^4$ in $[V_1]$. Let $[V_4] = \{\ H_4\ \}$, where H_4 is the graph induced by x_1,x_2,x_3 and x_4, $d_{H_4}(x_1) = 3$, $d_{H_4}(x_2)\ =\ d_{H_4}(x_3) = d_{H_4}(x_4) = 1$ in LTQ_1^4. Then $|\bigcup_{i=1}^4$ $N(x_i)\ \cap LTQ_3^4| = 4$. Hence $\bigcup_{i=1}^4 N(x_i)\ \cap LTQ_3^4$ is not in $[V_4]$. This implies $\bigcup_{i=1}^4\ N(x_i)\ \cap LQ_2^4$ in $[V_2]$. Consider the subgraph induced by $V(LTQ_3^4) \bigcup V(LTQ_4^4)$. Let $[V_1] = \{\ H_1\ \}$, where H_1: u_1,u_2,u_3 and u_4, $d_{H_1}(u_1) = 3$, $d_{H_1}(u_2) = d_{H_1}(u_3) = d_{H_1}(u_4) = 1$ in LTQ_3^4. Then $|\bigcup_{i=1}^4\ N(u_i)\ \cap LTQ_4^4| = 4$. Hence $\bigcup_{i=1}^4\ N(u_i)\ \cap LTQ_4^4$ is not in $[V_1]$. This implies $\bigcup_{i=1}^4\ N(u_i)\ \cap LTQ_4^4$ in $[V_2]$. Let $[V_2] = \{\ H_2$ $\}$, where H_2: v_1,v_2,v_3 and v_4, $d_{H_2}(v_1) = 3$, $d_{H_2}(v_2) = d_{H_2}(v_3) = d_{H_2}(v_4) = 1$ in LQ_3^4. Then $|\bigcup_{i=1}^4$ $N(v_i)\ \cap LTQ_4^4| = 4$. Hence $\bigcup_{i=1}^4\ N(v_i)\ \cap LTQ_4^4$ is not in $[V_2]$. This implies $\bigcup_{i=1}^4\ N(v_i)\ \cap LTQ_4^4$ in $[V_1]$. Let $[V_3] = \{w_1,w_2,w_3,w_4\}$ $[V_3] = \{x_1,x_2,x_3,x_4\}$. Since $|\bigcup_{i=1}^4\ N(w_i)\ \cap LTQ_4^4| = 0$ and $|\bigcup_{i=1}^4\ N(x_i)$

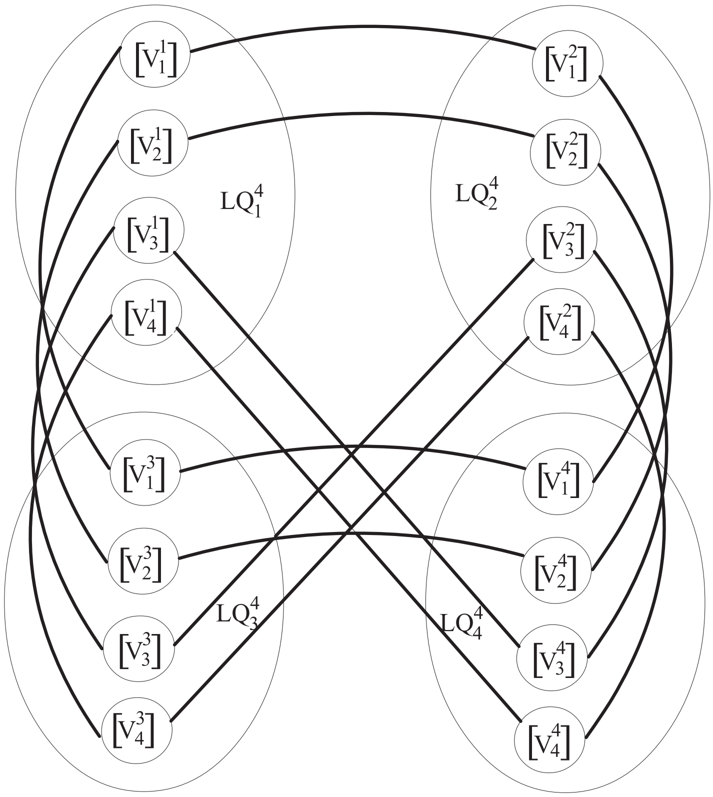

Figure 13. Possibility of edges in LTQ^4.

$\cap LTQ_4^4| = 0$ the role of the partition sets $[V_3]$ and $[V_4]$ in LTQ_3^4 is the same as that of $[V_3]$ and $[V_4]$ in LTQ_4^4.

$[V_1]$, $[V_2]$, $[V_3]$ and $[V_4]$ yield the optimal induced $K_{1,3}$-packing number of LTQ^6. This implies $ipp(LTQ^6)\ i \geq 4$. Assume that the result is true for LTQ^r. LTQ^r contains two copies of LTQ^{r-1}, say LTQ_1^{r-1} and Q_2^{r-1}. By the induction hypothesis $ipp(LTQ_1^{r-1})$ and $ipp(LTQ_1^{r-1})$ are 4. The union is an optimal induced $K_{1,3}$-packing number of LTQ^r. See Figure 13. ∎

Theorem 3.10: *For any positive integer m, there exists a connected graph G such that* $ipp(G, C_4) = m$.

Proof: Let K_m be the complete graph on m vertices. With every vertex of K_m, identify with an induced cycle of length 4 to obtain a new graph G on $4m$ vertices, where $V(G^*) = V(G) \cup (\cup_{i=1}^{m} \{ a_i', a_i'', a_i''' \})$ and $E(G^*) = E(G) \bigcup (\cup_{i=1}^{m} \{ (a_i, a_i'), (a_i'\ a_i''), (a_i''\ a_i'''), (a_i'''\ a_i)\})$. See Figure 14. We claim that $ipp(G, C_4) = m$. The collection of all induced C_4-cycles gives a perfect C_4-packing of G. No two vertex disjoint C_4-cycles can be in the same partition set, as there are two edges between them in G. Therefore the induced C_4-packing partition number of G is m. ∎

Theorem 3.11: *Let $K_{m,n}$ be the complete bipartite graph with a perfect or almost perfect induced C_4-packing. Then the induced C_4-packing k-partition number is* $\lfloor \frac{m+n}{4} \rfloor$.

Proof: Let X and Y be the bipartite sets of $K_{m,n}$ with $|X| = m$ and $|X| = n$. No two vertex disjoint C_4-cycles in $K_{m,n}$ can be in the same partition set, as there are at least 4 edges between them. Therefore the induced C_4-packing partition number of $K_{m,n}$ is $\lfloor \frac{m+n}{4} \rfloor$. See Figure 15. ∎

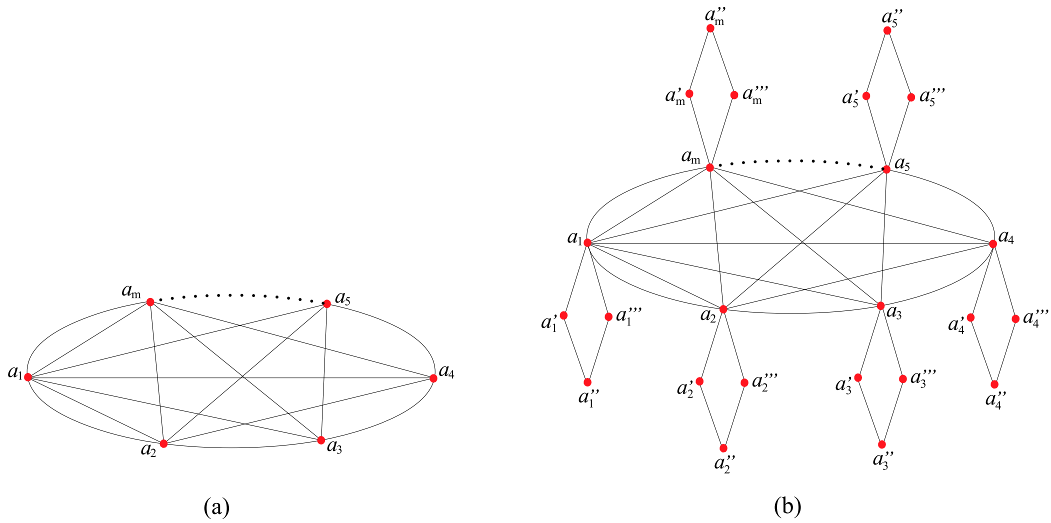

Figure 14. (a) Complete graph K_m (b)G G defined in Theorem 3.10.

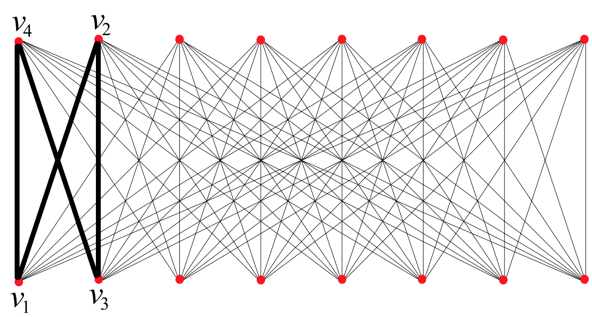

Figure 15. Complete bipartite graph $K_{8,8}$.

4. Conclusion

In this paper we have proved that the induced P_3-packing k-partition number for trees with perfect P_3-packing is 2. Further we have determined the induced P_3-packing k-partition number for slim trees, split graphs, complete bipartite graphs, grids and circulant graphs. We also deal with networks having perfect $K_{1,3}$-packing where $K_{1,3}$ is a claw on four vertices. We have proved that an induced $K_{1,3}$-packing k-partition problem is NP-Complete. Further we prove that induced $K_{1,3}$-packing k-partition of Q^r is 2 for all hypercube networks with perfect $K_{1,3}$-packing. We also obtained that $ipp(LQ^r) = 4$ for all locally twisted cubes with perfect $K_{1,3}$-packing. It would be an interesting line of research to determine the induced H-packing k-partition problem for other interconnection networks and Chemical graphs.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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