

# Total Chromatic Number of Star and BistarGraphs

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## Abstract

A total coloring of a graph  $G$  is an assignment of colors to both the vertices and edges of  $G$ , such that no two adjacent or incident vertices and edges of  $G$  are assigned the same colors. In this paper, we have discussed the total coloring of  $M(K_{1,n})$ ,  $T(K_{1,n})$ ,  $L(K_{1,n})$  and  $B_{n,n}^2$  and we obtained the total chromatic number of  $M(K_{1,n})$ ,  $T(K_{1,n})$ ,  $L(K_{1,n})$  and  $B_{n,n}^2$ .

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**Keywords and Phrases:** Middle graph, line graph, total graph, star graph, square graph, total coloring and total chromatic number.

## 1. Introduction

In this paper, we have chosen finite, simple and undirected graphs. Let  $G = (V(G), E(G))$  be a graph with the vertex set  $V(G)$  and the edge set  $E(G)$ , respectively. In 1965, the concept of total coloring was introduced by Behzad [1] and in 1967 he [2] came out new ideology that, the total chromatic number of complete graph and complete bi-partite graph. A total coloring of  $G$ , is a function  $f: S \rightarrow C$ , where  $S = V(G) \cup E(G)$  and  $C$  is a set of colors to satisfies the given conditions.

- (i) no two adjacent vertices receive the same colors
- (ii) notwo adjacent edges vertices receive the same colors
- (iii) no edges and its end vertices receive the same colors

The *total chromatic number*  $\chi''(G)$  of a graph  $G$  is the minimum cardinality  $k$  such that  $G$  may have a total coloring by  $k$  colors. Behzad [1] and Vizing [10] conjectured that for every simple graph  $G$  has  $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$ , where  $\Delta(G)$  the maximum degree of  $G$ . This conjecture is called the Total Coloring Conjecture (TCC). Rosenfeld [6] and Vijayaditya [9] verified the TCC, for any graph  $G$  with maximum degree  $\leq 3$  and Kostochka [4] for maximum degree  $\leq 5$ .

In Borodin [3] verified the total coloring conjecture (TCC) for maximum degree  $\geq 9$  in planar graphs. In 1992, Yap and Chew [11] proved that any graph  $G$  has a total coloring with at most  $\Delta(G) + 2$  colors if  $\Delta(G) \geq |V(G)| - 5$ . In recent era, total coloring have been extensively studied in different families of graphs. Mohan et.al [5] given the tight bound of Behzad and Vizing conjecture for Corona product of certain classes of graph. S.K. Vaidya et.al [8] prove that the total chromatic number of middle graph, total graph, shadow graph of cycle and one point union of cycle. S. Sudha et.al [7] prove that the total chromatic number of Sudha grid graphs, gear graph and crown graph. In this present paper, we investigate the total chromatic number of  $M(K_{1,n})$ ,  $T(K_{1,n})$ ,  $L(K_{1,n})$  and  $B_{n,n}^2$ .

## 2. Total Chromatic Number of $M(K_{1,n})$

**Definition 2.1.** The middle graph of a graph  $G$ , denoted by  $M(G)$  is define as follows, the vertex set of  $M(G)$  is  $V(G) \cup E(G)$ . Two vertices  $x, y$  in the vertex set of  $M(G)$  are adjacent in  $M(G)$  in case one of the following condition holds: (i)  $x, y$  are in  $E(G)$  and  $x, y$  is adjacent in  $G$ . (ii)  $x$  is in  $V(G)$ ,  $y$  is in  $E(G)$  and  $x, y$  are incident in  $G$ .

**Theorem 2.2.** Let  $M(K_{1,n})$  be the middle graph of star graph. Then

$$\chi''(M(K_{1,n})) = \begin{cases} n + 2, & n \text{ is odd} \\ n + 3, & n \text{ is even} \end{cases}$$

**Proof:**

Let  $V(K_{1,n}) = \{v\} \cup \{v_1, v_2, v_3, \dots, v_n\}$  and  $E(K_{1,n}) = \{e_1, e_2, e_3, \dots, e_n\}$  where  $\{e_i = vv_i : 1 \leq i \leq n\}$ . Now we construct the middle graph of star, each edge  $\{e_i = vv_i : 1 \leq i \leq n\}$  in  $K_{1,n}$  is subdivided by the vertices  $\{v'_i : 1 \leq i \leq n\}$  in  $M(K_{1,n})$  and the vertices  $\{v, v'_1, v'_2, v'_3, \dots, v'_n\}$  induce a clique of order  $n + 1$ . Let it be  $K_{n+1}$ . In  $M(K_{1,n})$ , the vertex set and the edge set is given by  $V(M(K_{1,n})) = \{v\} \cup \{v_1, v_2, v_3, \dots, v_n\} \cup \{v'_1, v'_2, v'_3, \dots, v'_n\}$  and  $E(M(K_{1,n})) = \{vv'_i : 1 \leq i \leq n\} \cup \{v'_i v'_j : 1 \leq i \leq n - 1, j > i, i + 1 \leq j \leq n\}$ . Now we define the total coloring  $f$ , such that  $f: S \rightarrow C$  as follows, where  $S = V(M(K_{1,n})) \cup E(M(K_{1,n}))$  and  $C = \{1, 2, 3, \dots, n + 3\}$ . The total coloring is obtained by coloring these vertices and edges as follows. We consider the following two cases:

**Case(i):** when  $n$  is odd

$$\begin{aligned}
 f(v) &= n + 2 & f(v'_i) &= i & : & 1 \leq i \leq n. \\
 f(v_i) &= n + 1 & : & 1 \leq i \leq n \\
 f(vv'_i) &= \begin{cases} 2i, & \text{if } 2i \not\equiv 0 \pmod{n + 2} \\ n + 2, & \text{otherwise} \end{cases} & \text{for } & 1 \leq i \leq n \\
 f(v'_i v'_j) &= \begin{cases} i + j, & \text{if } (i + j) \not\equiv 0 \pmod{n + 2} \\ n + 2, & \text{otherwise} \end{cases} & & \\
 & & & \text{for } & 1 \leq i \leq n - 1, j > i, i + 1 \leq j \leq n \\
 f(v'_i v_i) &= i - 1 & : & 2 \leq i \leq n \\
 f(v'_1 v_1) &= n + 2
 \end{aligned}$$

Based on the above method of total coloring, the graph  $M(K_{1,n})$  is total colored with  $n + 2$  colors.

Hence the total chromatic number of  $M(K_{1,n})$ ,  $\chi''(M(K_{1,n})) = n + 2$ .

**Case (ii):** when  $n$  is even

$$\begin{aligned}
 f(v) &= n + 3 & f(v'_i) &= i & : & 1 \leq i \leq n. \\
 f(v_i) &= n + 1 & : & 1 \leq i \leq n \\
 f(vv'_i) &= \begin{cases} 2i, & \text{if } 2i \not\equiv 0 \pmod{n + 3} \\ n + 3, & \text{otherwise} \end{cases} & \text{for } & 1 \leq i \leq n \\
 f(v'_i v'_j) &= \begin{cases} i + j, & \text{if } (i + j) \not\equiv 0 \pmod{n + 3} \\ n + 3, & \text{otherwise} \end{cases} & & \\
 & & & \text{for } & 1 \leq i \leq n - 1, j > i, i + 1 \leq j \leq n \\
 f(v'_i v_i) &= i - 1 & : & 2 \leq i \leq n \\
 f(v'_1 v_1) &= n + 2
 \end{aligned}$$

Based on the above rule of total coloring, the graph  $M(K_{1,n})$  is total colored with  $n + 3$  colors. Hence the total chromatic number of  $M(K_{1,n})$ ,  $\chi''(M(K_{1,n})) = n + 3$ .

**Illustration 2.3:** Consider the middle graph of star

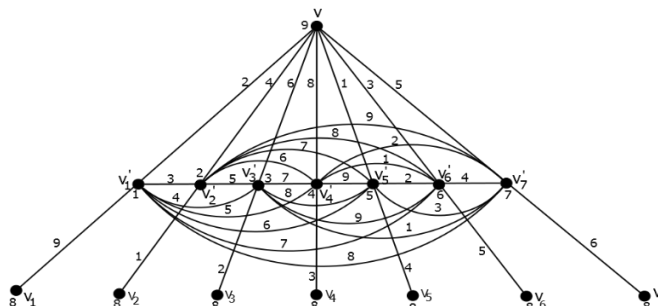


Figure 1: Total Coloring of Middle Graphof Star  $M(K_{1,7})$  for n is Odd

**Illustration 2.4:** Consider the middle graph of star

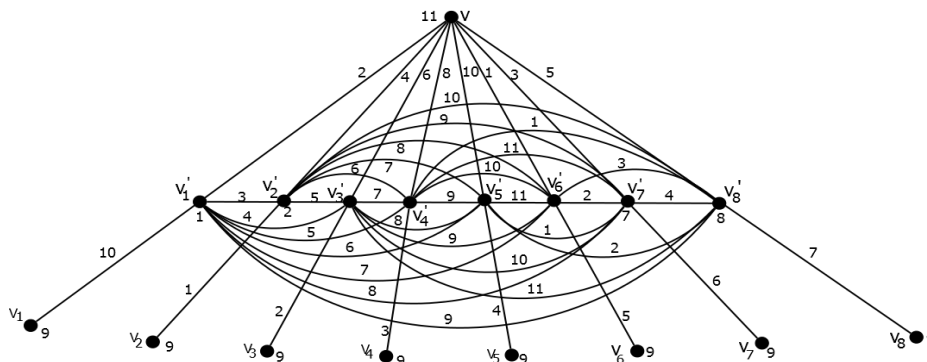


Figure 2: Total Coloring of Middle Graph of Star  $M(K_{1,8})$ for n is Even

### 3. Total Chromatic Number of $T(K_{1,n})$

**Definition 3.1.** The *Total graph* of a graph  $G$ , denoted by  $T(G)$  is define as, the vertex set of  $T(G)$  is  $V(G) \cup E(G)$ . Two vertices  $x, y$  in the vertex set of  $T(G)$  are adjacent in  $T(G)$  in case one of the following condition holds: (i)  $x, y$  are in  $V(G)$  and  $x$  is adjacent to  $y$  in  $G$ . (ii)  $x, y$  are in  $E(G)$  and  $x, y$  is adjacent in  $G$ . (iii)  $x$  is in  $V(G), y$  is in  $E(G)$  and  $x, y$  are incident in  $G$ .

**Theorem 3.2.** Let  $T(K_{1,n})$  be the total graph of star graph. Then

$$\chi^n(T(K_{1,n})) = 2n + 1,$$

**Proof:**

Let  $V(K_{1,n}) = \{v\} \cup \{v_1, v_2, v_3, \dots, v_n\}$  and  $E(K_{1,n}) = \{e_1, e_2, e_3, \dots, e_n\}$  where  $\{e_i = vv_i : 1 \leq i \leq n\}$ . Now we construct the total graphof star, each edge  $\{e_i = vv_i : 1 \leq i \leq n\}$  in  $K_{1,n}$  is subdivided by the vertices  $\{v'_i : 1 \leq i \leq n\}$  in  $T(K_{1,n})$  and the vertices  $\{v, v'_1, v'_2, v'_3, \dots, v'_n\}$  induce a clique of order  $n + 1$ . Let it be  $K_{n+1}$ .

In  $T(K_{1,n})$ , the vertex set and the edge set is given by  $V(T(K_{1,n})) = \{v\} \cup \{v_1, v_2, v_3, \dots, v_n\} \cup \{v'_1, v'_2, v'_3, \dots, v'_n\}$  and  $E(T(K_{1,n})) = \{vv'_i : 1 \leq i \leq n\} \cup \{vv_i : 1 \leq i \leq n\} \cup \{v'_i v'_j : 1 \leq i \leq n-1, j > i, i+1 \leq j \leq n\}$ .

Now we define the total coloring  $f$ , such that  $f: S \rightarrow C$  as follows, where  $S = V(T(K_{1,n})) \cup E(T(K_{1,n}))$  and  $C = \{1, 2, 3, \dots, 2n + 1\}$ . The total coloring is obtained by coloring these vertices and edges as follows.

$$\begin{aligned}
 f(v) &= 2n + 1 & f(v'_i) &= i & : & 1 \leq i \leq n. \\
 f(v_i) &= 2n & & & & : 1 \leq i \leq n \\
 f(vv'_i) &= \begin{cases} 2i, & \text{if } 2i \not\equiv 0 \pmod{2n} \\ 2n, & \text{Otherwise} \end{cases} & & & & \text{for } 1 \leq i \leq n \\
 f(vv_i) &= \begin{cases} 2i - 1, & \text{if } 2i - 1 \not\equiv 0 \pmod{2n - 1} \\ 2n - 1, & \text{Otherwise} \end{cases} & & & & \text{for } 1 \leq i \leq n \\
 f(v'_i v'_j) &= \begin{cases} i + j, & \text{if } (i + j) \not\equiv 0 \pmod{2n + 1} \\ 2n + 1, & \text{otherwise} \end{cases} & & & & \text{for } 1 \leq i \leq n - 1, j > i, i + 1 \leq j \leq n \\
 f(v'_i v_i) &= i - 1 & : & 2 \leq i \leq n \\
 f(v_1 v_1) &= n + 2
 \end{aligned}$$

Based on the above rule of total coloring, the graph  $T(K_{1,n})$  is total colored with  $2n + 1$  colors. Hence the total chromatic number of the total graph of  $T(K_{1,n})$ ,  $\chi''(T(K_{1,n})) = 2n + 1$ .

**Illustration 3.3:** Consider the total graph of Star graph

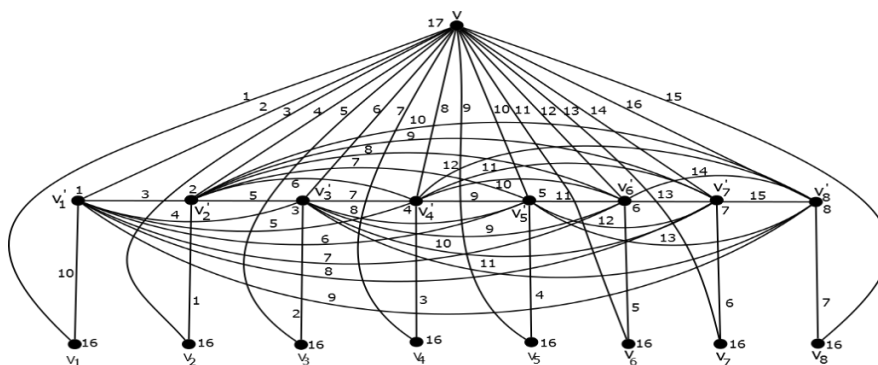


Figure 3: Total Coloring of Total Graph of Star  $T(K_{1,8})$

### 4. Total Chromatic Number of $L(B_{n,n})$

**Definition 4.1.** *Bistar graph* is the graph obtained by joining the root vertices of two copies of star  $K_{1,n}$

**Definition 4.2.** The *line graph* of graph  $G$ , denoted by  $L(G)$  is the graph whose vertices are the edges of  $G$ , (iii)  $x$  is in  $V(G)$ ,  $y$  is in  $E(G)$  and  $x, y$  are incident in  $G$ .

**Theorem4.3.** Let  $L(B_{n,n})$  be the line graph of star graph. Then

$$\chi''(L(B_{n,n})) = 2n + 1,$$

**Proof:**

Let  $V(B_{n,n}) = \{u\} \cup \{v\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\}$  and  $E(B_{n,n}) = \{uu_i : 1 \leq i \leq n\} \cup \{vv_i : 1 \leq i \leq n\} \cup \{uv\}$ . Let  $u'_i (1 \leq i \leq n)$  be the edge between the vertices  $u$  and  $u_i (1 \leq i \leq n)$  (i.e)  $uu_i = u'_i$  and  $v'_i (1 \leq i \leq n)$  be the edge between the vertices  $v$  and  $v_i (1 \leq i \leq n)$  (i.e)  $vv_i = v'_i$  and  $u'$  be the edge between the vertices  $u$  and  $v$ . Here  $u'$  is adjacent with both the vertices  $u_i (1 \leq i \leq n)$  and  $v_i (1 \leq i \leq n)$ . Now we construct the line graph of Bistar, the edge set of  $B_{n,n}$  corresponding to the vertex set of  $L(B_{n,n})$ . The vertices  $u'_i (1 \leq i \leq n)$  along with  $u'$  forms a complete graph of order  $n + 1$  and also the vertices  $v'_i (1 \leq i \leq n)$  along with  $u'$  forms a complete graph of order  $n + 1$ . Let it be  $K'_{n+1}$  and  $K''_{n+1}$ . In  $L(B_{n,n})$ , the vertex set and the edge set is given by

$$V(L(B_{n,n})) = \{u'_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \cup \{u'\} \text{ and}$$

$$E(L(B_{n,n})) = \{u'u'_i : 1 \leq i \leq n\} \cup \{u'v'_i : 1 \leq i \leq n\}$$

$$\cup \{u'_i u'_j : 1 \leq i \leq n - 1, j > i, i + 1 \leq j \leq n\} \cup$$

$$\{v'_i v'_j : 1 \leq i \leq n - 1, j > i, i + 1 \leq j \leq n\}.$$

Now we construct the total coloring  $f$ , such that  $f: S \rightarrow C$  as follows, where  $S = V(L(B_{n,n})) \cup E(L(B_{n,n}))$  and  $C = \{1, 2, 3, \dots, 2n + 1\}$ . The total coloring is obtained by coloring these vertices and edges as follows.

$$f(u') = 2n + 1$$

$$f(u'_i) = \begin{cases} i, & \text{if } i \not\equiv 0 \pmod{n} \\ n, & \text{otherwise} \end{cases} \text{ for } 1 \leq i \leq n$$

$$f(v'_i) = \begin{cases} n + i, & \text{if } n + i \not\equiv 0 \pmod{2n} \\ 2n, & \text{otherwise} \end{cases} \text{ for } 1 \leq i \leq n$$

$$f(u'u'_i) = \begin{cases} 2i, & \text{if } 2i \not\equiv 0 \pmod{2n} \\ 2n, & \text{otherwise} \end{cases} \text{ for } 1 \leq i \leq n$$

$$f(u'v'_i) = \begin{cases} 2i - 1, & \text{if } 2i - 1 \not\equiv 0 \pmod{2n - 1} \\ 2n - 1, & \text{otherwise} \end{cases} \text{ for } 1 \leq i \leq n$$

$$f(u'_i u'_j) = \begin{cases} i + j, & \text{if } (i + j) \not\equiv 0 \pmod{2n + 1} \\ 2n + 1, & \text{otherwise} \end{cases}$$

$$f(v'_i v'_j) = \begin{cases} i + j - 1, & \text{if } (i + j - 1) \not\equiv 0 \pmod{2n + 1} \\ 2n + 1, & \text{otherwise} \end{cases}$$

for  $1 \leq i \leq n - 1, j > i, i + 1 \leq j \leq n$   
for  $1 \leq i \leq n - 1, j > i, i + 1 \leq j \leq n$

It is clear that, the above rule of total coloring, the graph  $L(B_{n,n})$  is total colored with  $2n + 1$  colors.

Hence the total chromatic number of  $L(B_{n,n}), \chi''(L(B_{n,n})) = 2n + 1$ .

**Illustration 4.4:** Consider the line graph of Bistar graph

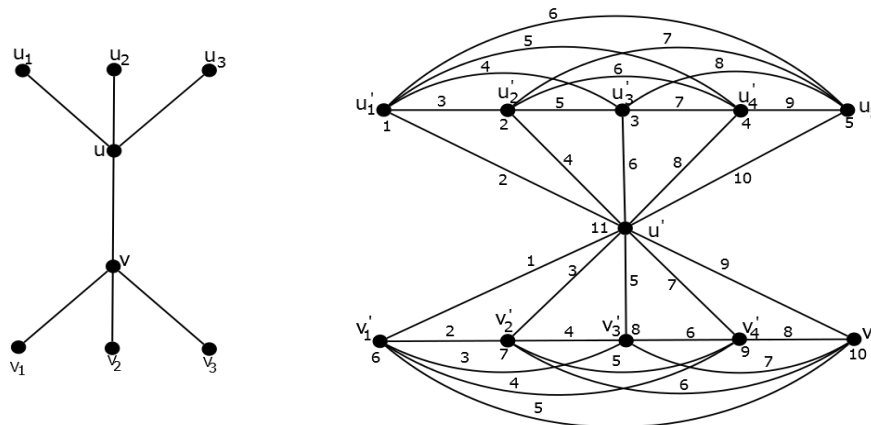


Figure 4: Bistar Graph  $B_{3,3}$  and Total Coloring of Line Graph of Bistar  $L(B_{5,5})$

### 5. Total Chromatic Number of $B_{n,n}^2$

**Definition 5.1.** For a simple connected graph  $G$ , the square graph  $G^2$  is the graph  $G^2$  obtained by adding edges to  $G$  between any two vertices of  $G$  is distance 2.

**Theorem 5.2.** Let  $B_{n,n}^2$  be the square graph of Bistar graph. Then

$$\chi''(B_{n,n}^2) = 2n + 2, \quad n \geq 2.$$

**Proof:**

Let  $V(B_{n,n}) = \{u\} \cup \{v\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\}$  and

$E(B_{n,n}) = \{uu_i : 1 \leq i \leq n\} \cup \{vv_i : 1 \leq i \leq n\} \cup \{uv\}$ . Now we construct the square graph of Bistar graph, the new edges in  $B_{n,n}^2$  is obtained by adding the edges to  $B_{n,n}$  between any two vertices of  $B_{n,n}$  of distance two. In  $B_{n,n}^2$ , the vertex set and the edge set is given by

$$V(B_{n,n}^2) = \{u\} \cup \{v\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \text{ and}$$

$$E(B_{n,n}^2) = \{uu_i : 1 \leq i \leq n\} \cup \{uv_i : 1 \leq i \leq n\} \cup \{vu_i : 1 \leq i \leq n\} \cup \{vv_i : 1 \leq i \leq n\}$$

Now we construct the total coloring  $f$ , such that  $f: S \rightarrow C$  as follows, where  $S = V(B_{n,n}^2) \cup E(B_{n,n}^2)$  and  $C = \{1, 2, 3, \dots, 2n + 2\}$ . The total coloring is obtained by coloring these vertices and edges as follows.

$$f(u) = 1 \quad f(v) = 2n + 1$$

$$f(u_i) = f(v_i) = f(uv) = 2n + 2 : 1 \leq i \leq n$$

$$\begin{aligned}
 f(uu_i) &= \begin{cases} 2i, & \text{if } 2i \not\equiv 0 \pmod{2n} \\ 2n, & \text{Otherwise} \end{cases} \text{ for } 1 \leq i \leq n \\
 f(uv_i) &= \begin{cases} 2i + 1, & \text{if } 2i + 1 \not\equiv 0 \pmod{2n + 1} \\ 2n + 1, & \text{Otherwise} \end{cases} \text{ for } 1 \leq i \leq n \\
 f(vu_i) &= \begin{cases} 2i - 1, & \text{if } 2i - 1 \not\equiv 0 \pmod{2n - 1} \\ 2n - 1, & \text{Otherwise} \end{cases} \text{ for } 1 \leq i \leq n \\
 f(vv_i) &= \begin{cases} 2i, & \text{if } 2i \not\equiv 0 \pmod{2n} \\ 2n, & \text{Otherwise} \end{cases} \text{ for } 1 \leq i \leq n
 \end{aligned}$$

It is clear that, the above rule of total coloring, the graph  $B_{n,n}^2$  is total colored with  $2n + 2$  colors. Hence the total chromatic number of  $B_{n,n}^2$ ,  $\chi''(B_{n,n}^2) = 2n + 2$ .

**Illustration 5.3:** Consider the square graph of Bistar graph.

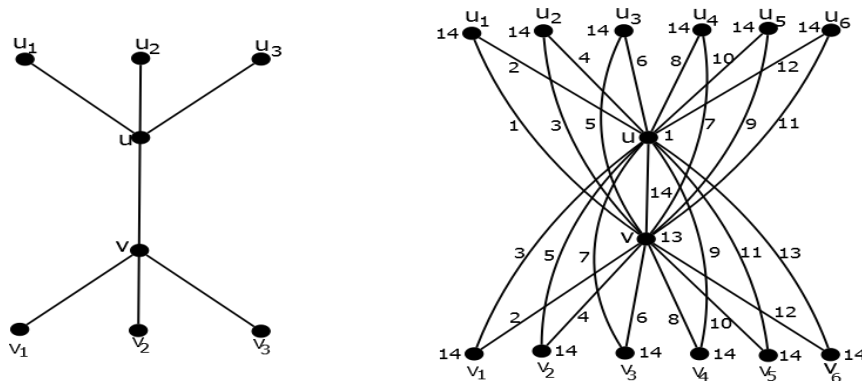


Figure 5: Bistar Graph  $B_{3,3}$  and Total Coloring of Square Graph of Bistar  $B_{6,6}^2$

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