Total Coloring of Middle, Total graph of Bistar, Double wheel and Double Crown Graph

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Abstract

A *total coloring* of a graph G is an assignment of colors to both the vertices and edges of G, such that no two adjacent or incident vertices and edges of G are assigned the same colors. In this paper, we have discussed the total coloring of $M(B_{n,n}), T(B_{n,n}), DW_n$ and C_n^{++} and also we obtained the total chromatic number of $M(B_{n,n}), T(B_{n,n}), DW_n$ and C_n^{++}

AMS Subject Classification: 05C15

Keywords : Middle graph, total graph, double wheel, double crown and total chromatic number.

1. INTRODUCTION

All graphs consider here finite, simple and undirected graphs. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$, and the edge set $E(G)$, respectively. A coloring of a graph G is an assignment of colors to the vertices or edges or both. A vertex-coloring(edge-coloring) is called proper coloring if no two vertices(edges) receive the color. There are so many different proper colorings such as a-coloring, b-coloring, star coloring, list coloring, harmonious coloring, total coloring etc. In the present work focused on total coloring of graphs.

A total coloring of G, is a function $f : S \to C$, where $S = V(G) \cup E(G)$ and C is a set of colors to satisfies the given conditions.

(i) no two adjacent vertices receive the same colors

(ii) no two adjacent edges receive the same colors

(iii) no edges and its end vertices receive the same colors

The concept of total coloring was introduced by Behzad[1] and Vizing[10]. Also they have posed the conjecture that for every simple graph G has $\Delta(G)+1 \le \chi''(G) \le \Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G. This conjecture is known as the Total Coloring Conjecture

(TCC). The total chromatic number $\chi''(G)$ of a graph G is the minimum cardinality k such that G may have a total coloring by k colors. Behzad *et .al* [2] proved that the total chromatic number of complete graph. Rosenfeld[8] and Vijayaditya[9] verified the TCC, for any graph G with maximum degree ≤ 3 and Kostochka[6] for maximum degree ≤ 5. In Borodin[4] verified The Total Coloring conjecture (TCC) for maximum degree ≥ 9 in planar graphs. In recent era, total coloring have been extensively studied in different families of graphs. Mohan et.al[7] given the tight bound of Behzad and Vizing conjecture in Corona product of certain classes of graph. Jayaraman et.al[5] proved that the total chromatic number of double star graph families. In the present work, we investigate the total chromatic number of $M(B_{n,n})$, $T(B_{n,n})$, DW_n and C_n^{++}. The *Middle graph*[5] of a graph G, denoted by $M(G)$ is define as follows, the vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following condition holds: (i) x, y are in $E(G)$ and x, y is adjacent in G (ii) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G. The *Total graph*[3] of a graph G, denoted by $T(G)$ is define as, the vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following condition holds: (i) x, y are in $V(G)$, and x is adjacent to y in G (ii) x, y are in $E(G)$ and x, y is adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G. The *Double Crown graph* of a graph G, denoted by C_n^{++}, is the graph obtained from the cycle C_n by attaching two pendent edges at each vertex of G. The *Double wheel graph*, denoted by DW_n, is the graph obtained by joining all vertices of a disjoint union of two cycles C_n to an external vertex. That is, $2C_n + K_1$. In this paper, we have discussed the total coloring of $M(B_{n,n}), T(B_{n,n}), DW_n$ and C_n^{++} and also we obtained the total chromatic number of $M(B_{n,n}), T(B_{n,n}), DW_n$ and C_n^{++}.

2. MAIN RESULTS

Theorem.2.1. *Let $M(B_{n,n})$ be the middle graph of bistar graph. Then* $\chi''(M(B_{n,n})) = 2n+3$

Proof: Let $V(B_{n,n}) = \{u\} \bigcup \{v\} \bigcup \{u_i : 1 \leq i \leq n\} \bigcup \{v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uv\} \bigcup \{uu_i, vv_i : 1 \leq i \leq n\}$. By the definition of middle graph, introduce a new vertices $\{u_i' : 1 \leq i \leq n\}$ and $\{v_i' : 1 \leq i \leq n\}$ in the edge connecting $\{uu_i : 1 \leq i \leq n\}$ and $\{vv_i : 1 \leq i \leq n\}$ and u' be the newly introduced vertex between u and v. In $M(B_{n,n})$, the vertex set and the edge set is given by

$$V(M(B_{n,n})) = \{u\} \bigcup \{v\} \bigcup \{u'\} \bigcup \{u_i, v_i, u_i', v_i' : 1 \leq i \leq n\}$$

Here $\{u_i' : 1 \leq i \leq n\}$ along with the vertex u and u' induces a clique of order $n+2$, also $\{v_i' : 1 \leq i \leq n\}$ along with the vertex v and u' induces an another clique of order $n+2$. Thus we see that there are two edge disjoint sub graph of order K_{n+2}

$$E\big(M(B_{n,n})\big)=\begin{cases}\{u'v_i',vv_i',v_i'v_i,u'u_i',uu_i',u_i'u_i:1\le i\le n\}\cup\\ \{uu'\}\cup\{u'v\}\cup\{u_i'u_j',v_i'v_j':i+1\le i\le n,j>i\}\end{cases}$$

We define the total coloring f, such that $f:S\to C$, where $S=V\big(M(B_{n,n})\big)\cup E\big(M(B_{n,n})\big)$ and $C=\{1,2,3,.....2n+3\}$ be the set of colors. Now we assign the total coloring to these vertices and edges as follows.

$$f(u)=2n+1 \qquad f(u')=n+1 \qquad f(v)=2n+2$$

$$f(uu')=2n+2 \qquad f(u'v)=2n+3$$

$$f(u_i)=f(v_i)=2n+3,\quad for\ 1\le i\le n$$

$$f(u_i')=f(u'v_i')=i,\quad f(v_i')=f(uu_i')=2i,\quad for\ 1\le i\le n$$

$$f(u_i'u_j')=f(v_i'v_j')=\begin{cases}i+j, & if\,(i+j)\ne 0(\bmod 2n+3),\ for\ 1\le i\le n\\ 2n+3, & otherwise,\quad j>i,\quad i+1\le j\le n\end{cases}$$

$$f(u_iu_i')=f(v_iv_i')=\begin{cases}n+2+i, & if\,(n+2+i)\ne 0(\bmod 2n+3),\\ 2n+3, & otherwise,\end{cases}\quad for\ 1\le i\le n$$

$$f(u'u_i')=f(vv_i')=\begin{cases}n+1+i, & if\,(n+1+i)\ne 0(\bmod 2n+3),\\ 2n+3, & otherwise,\end{cases}\quad for\ 1\le i\le n$$

Using the above pattern of coloring, the graph $M(B_{n,n})$ is total colored with $2n+3$ colors. Hence $\chi''\big(M(B_{n,n})\big)=2n+3$.

Illustration: Total coloring of middle graph of bistar $M(B_{4,4})$ as shown in Figure 1

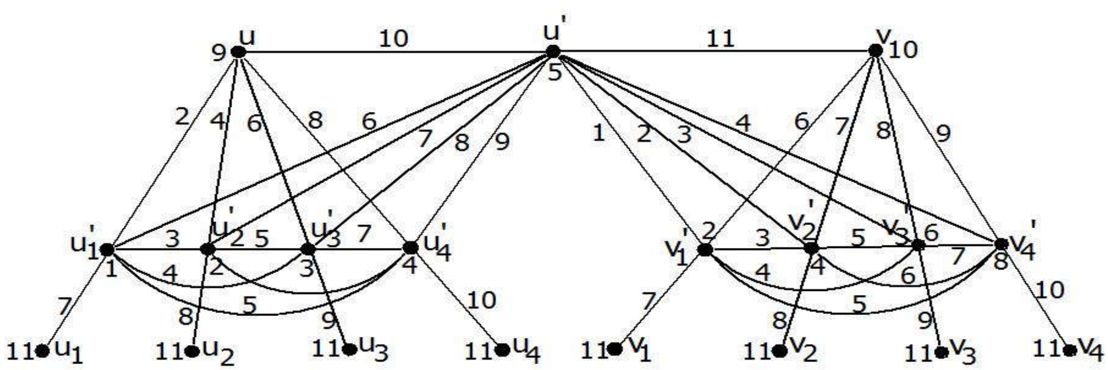

Figure 1: Total coloring of middle graph of bistar $M(B_{4,4})$

Theorem.2.2. *Let $T(B_{n,n})$ be the total graph of bistar graph. Then* $\chi''\big(T(B_{n,n})\big)=2n+3$.

Proof: Let $V(B_{n,n})=\{u\}\cup\{v\}\cup\{u_i:1\le i\le n\}\cup\{v_i:1\le i\le n\}$ and $E(B_{n,n})=\{uv\}\cup\{uu_i,vv_i:1\le i\le n\}$. By the definition of total graph, introduce a new vertices

$\{u_i^{'}:1 \le i \le n\}$ and $\{v_i^{'}:1 \le i \le n\}$ in the edge connecting $\{uu_i:1 \le i \le n\}$ and $\{vv_i:1 \le i \le n\}$ and $u^{'}$ be the newly introduced vertex between u and v. In $T(B_{n,n})$, the vertex set and the edge set is given by $V(T(B_{n,n})) = \{u\} \cup \{v\} \cup \{u^{'}\} \cup \{u_i, v_i, u_i^{'}, v_i^{'}:1 \le i \le n\}$

Here $\{u_i^{'}:1 \le i \le n\}$ along with the vertex u and $u^{'}$ induces a clique of order $n+2$, also $\{v_i^{'}:1 \le i \le n\}$ along with the vertex v and $u^{'}$ induces an another clique of order $n+2$. Thus we see that there are two edge disjoint sub graph of order K_{n+2}

$$E(T(B_{n,n})) = \begin{cases} \{uu_i, vv_i, u^{'}v_i^{'}, vv_i^{'}, v_i^{'}v_i, u^{'}u_i^{'}, uu_i^{'}, u_i^{'}u_i:1 \le i \le n\} \cup \\ \{uu^{'}\} \cup \{u^{'}v\} \cup \{uv\} \cup \{u_i^{'}u_j^{'}, v_i^{'}v_j^{'}: i+1 \le i \le n, j > i\} \end{cases}$$

We define the total coloring f, such that $f: S \to C$, where $S = V(T(B_{n,n})) \cup E(T(B_{n,n}))$ and $C = \{1,2,3,.....2n+3\}$ be the set of colors. Now we assign the total coloring to these vertices and edges as follows.

$$f(u) = 2n+1 \qquad f(u^{'}) = 2n+3 \qquad f(v) = 2n+2$$

$$f(u_i) = f(v_i) = 2n+3, \quad for\ 1 \le i \le n$$

$$f(u_i^{'}) = f(vv_i) = f(u^{'}v_i^{'}) = i, \quad f(v_i^{'}) = f(uu_i^{'}) = 2i, \quad for\ 1 \le i \le n$$

$$f(uu^{'}) = 2n+2, \qquad f(u^{'}v) = n+1, \qquad f(uv) = 2n+3$$

$$f(u_i^{'}u_j^{'}) = f(v_i^{'}v_j^{'}) = \begin{cases} i+j, & if\ (i+j) \ne 0 (\bmod 2n+3), \quad for\ 1 \le i \le n \\ 2n+3, & otherwise, \quad j > i, \quad i+1 \le j \le n \end{cases}$$

$$f(u_i u_i^{'}) = f(v_i v_i^{'}) = \begin{cases} n+2+i, & if\ (n+2+i) \ne 0 (\bmod 2n+3), \\ 2n+3, & otherwise, \end{cases} for\ 1 \le i \le n$$

$$f(u^{'}u_i^{'}) = f(vv_i^{'}) = \begin{cases} n+1+i, & if\ (n+1+i) \ne 0 (\bmod 2n+3), \\ 2n+3, & otherwise, \end{cases} for\ 1 \le i \le n$$

$$f(uu_i) = 2i-1, \quad for\ 1 \le i \le n$$

Using the above pattern of coloring, the graph $T(B_{n,n})$ is total colored with $2n+3$ colors. Hence $\chi''(T(B_{n,n})) = 2n+3$.

Illustration: Total coloring of total graph of bistar $T(B_{4,4})$ as shown in Figure 2

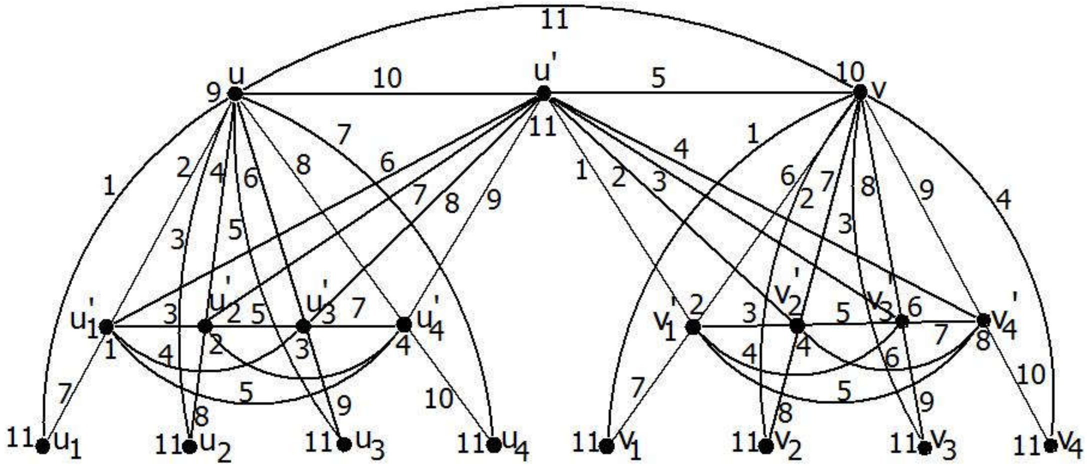

Figure 2: Total coloring of total graph of bistar $T(B_{4,4})$

Theorem 2.3. *Let* DW_n *be the Double wheel graph. Then* $\chi''(DW_n) = n+1, n \geq 4$

Proof: Let DW_n be the double wheel graph with $2n+1$ vertices and $2n$ edges. Let v_0 be the apex vertex. Let $\{u_1, u_2, \ldots u_n\}$ and $\{v_1, v_2, \ldots v_n\}$ be the rim vertices of DW_n. In DW_n, the vertex set and the edge set is given by $V(DW_n) = \{v_0\} \cup \{v_i, v_i^{'} : 1 \leq i \leq n\}$ and $E(DW_n) = \{e_i, e_i^{'}, e_i^{''}, e_i^{'''} : 1 \leq i \leq n\}$ where $e_i (1 \leq i \leq n)$ is an edge $v_0 v_i (1 \leq i \leq n), e_i^{'} (1 \leq i \leq n)$ is an edge $v_0 v_i^{'} (1 \leq i \leq n), e_i^{''} (1 \leq i \leq n-1)$, is an edge $v_i v_{i+1} (1 \leq i \leq n-1), e_i^{'''} (1 \leq i \leq n-1)$, is an edge $v_i^{'} v_{i+1}^{'} (1 \leq i \leq n-1), e_n^{''}$ is an edge $v_n v_1$ and $e_n^{'''}$ is an edge $v_n^{'} v_1^{'}$.

Now we construct the total coloring f, such that $f : S \to C$ as follows, where $S = V(DW_n) \cup E(DW_n)$ and $C = \{1, 2, 3 \ldots\ldots 2n+1\}$ be the set of colors. The total coloring is obtained by coloring these vertices and edges as follows.

$$f(v_0) = 2n+1, \quad f(v_i^{'}) = f(e_i) = i, \quad for\ 1 \leq i \leq n$$

$$f(v_i) = \begin{cases} i+1, & if\,(i+1) \neq 0 (\bmod 2n+1), \\ 2n+1, & otherwise, \end{cases} \quad for\ 1 \leq i \leq n$$

$$f(e_i^{'}) = f(e_i^{''}) = \begin{cases} n+i, & if\,(n+i) \neq 0 (\bmod 2n+1), \\ 2n+1, & otherwise, \end{cases} \quad for\ 1 \leq i \leq n$$

$$f(e_i^{'''}) = \begin{cases} i+2, & if\,(i+2) \neq 0 (\bmod 2n+1), \\ 2n+1, & otherwise, \end{cases} \quad for\ 1 \leq i \leq n-1$$

$$f(e_n^{'''}) = 2$$

It is clear that, the above method of total coloring, the graph DW_n is total colored with $2n+1$ colors. Hence $\chi''(DW_n) = 2n+1$ colors.

Illustration: Total coloring of double wheel graph DW_7 as shown in Figure 3.

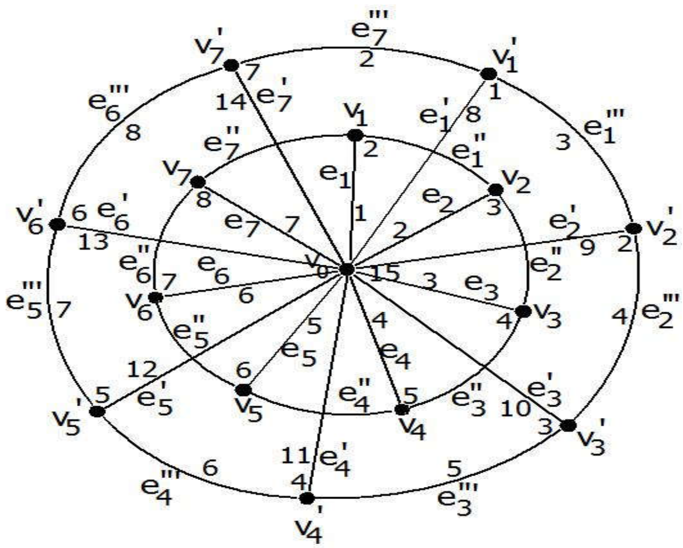

Figure 3: Total coloring of double wheel graph DW_7

Theorem 2.4 *Let C_n^{++} be the double crown graph. Then* $\chi''(C_n^{++}) = 5$

Proof: Let C_n^{++} be double crown graph with $3n$ vertices and $3n$ edges. The vertex set and edge set is given by $V(C_n^{++}) = \{v_i : 1 \le i \le n\} \cup \{v_{i,j} : 1 \le i \le n \;\; and \;\; j = 1,2\}$

$$E(C_n^{++}) = \{v_i v_{i+1} : 1 \le i \le n;\; i+1 \; taken \bmod n\} \cup \{e_{i,j} = (v_i, v_{i,j}) : 1 \le i \le n \;\; and \;\; j = 1,2\}$$

Now we construct the total coloring f, such that $f : S \to C$, where $S = V(C_n^{++}) \cup E(C_n^{++})$ and C is the set of colors. We assign the colors to these vertices and edges as follows. While coloring the value of $\bmod 2$ is equal to 0, it should be replace by 2. We consider the following two cases

Case(i): When n is even

$$f(v_i) = \begin{cases} 1, & if \;\; i \equiv 1 (\bmod 2), \\ 2, & if \;\; i \equiv 0 (\bmod 2), \end{cases} \;\; for \;\; 1 \le i \le n$$

$$f(v_i v_j) = 3, \quad for \;\; 1 \le i \le n \;\; and \;\; j = 1,2$$

$$f(v_i v_{i+1}) = \begin{cases} 3, & if\ i \equiv 1(\text{mod}\,2), \\ 4, & if\ i \equiv 0(\text{mod}\,2), \end{cases} \quad for\ \ 1 \le i \le n-1$$

$$f(v_n v_1) = 4, \quad f(e_{i,j}) = 5, \quad for\ 1 \le i \le n\ and\ \ j = 1$$

$$f(e_{i,j}) = \begin{cases} 2, & if\ i \equiv 1(\text{mod}\,2), \\ 1, & if\ i \equiv 0(\text{mod}\,2), \end{cases} \quad for\ \ 1 \le i \le n\ and\ \ j = 2$$

Illustration: Total coloring of double crown graph C_8^{++} as shown in Figure 4.

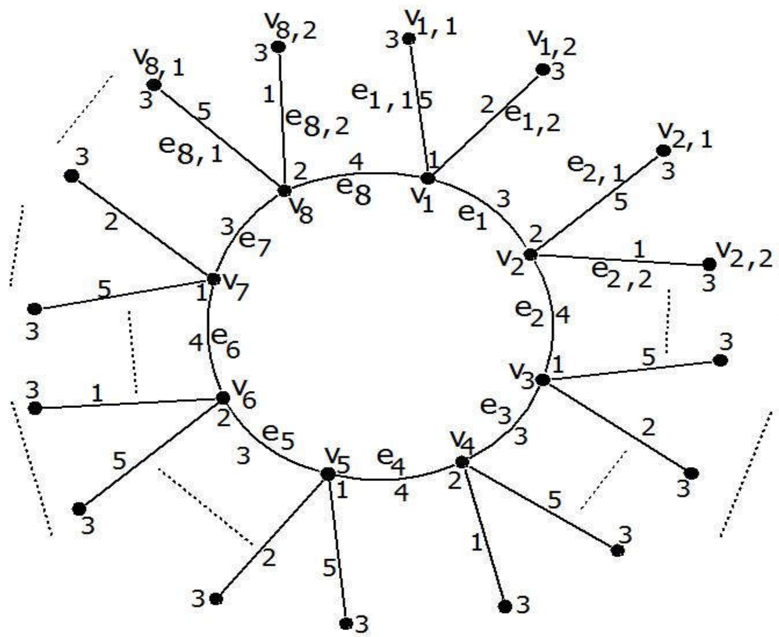

Figure 4: Total coloring of double crown graph C_8^{++}

Case(ii): When n is odd

$$f(v_i) = \begin{cases} 1, & if\ i \equiv 1(\text{mod}\,2), \\ 2, & if\ i \equiv 0(\text{mod}\,2), \end{cases} \quad for\ \ 1 \le i \le n-1$$

$$f(v_n) = 3, \qquad f(v_{i,j}) = 3, \quad for\ \ 1 \le i \le n-1 and\ \ j = 1, 2$$

$$f(v_{n,1}) = f(v_{n,2}) = 4$$

$$f(v_i v_{i+1}) = \begin{cases} 3, & if\ i \equiv 1(\text{mod}\,2), \\ 4, & if\ i \equiv 0(\text{mod}\,2), \end{cases} \quad for\ \ 1 \le i \le n-1$$

$$f(v_n v_1) = 5, \quad f(e_{i,j}) = 5, \quad for\ 2 \le i \le n-1 and\ \ j = 1$$

$$f(e_{1,1}) = 4, \qquad f(e_{n,1}) = 1$$

$$f(e_{i,j}) = \begin{cases} 2, & if\ i \equiv 1(\text{mod}\,2), \\ 1, & if\ i \equiv 0(\text{mod}\,2), \end{cases} \quad for\ 1 \le i \le n\ and\ j = 2$$

Based on the above procedure of the coloring pattern, we observe that, the graph C_n^{++} is total colored with 5 colors. Hence $\chi''(C_n^{++}) = 5$ colors.

Illustration: Total coloring of double crown graph C_7^{++} as shown in Figure 5.

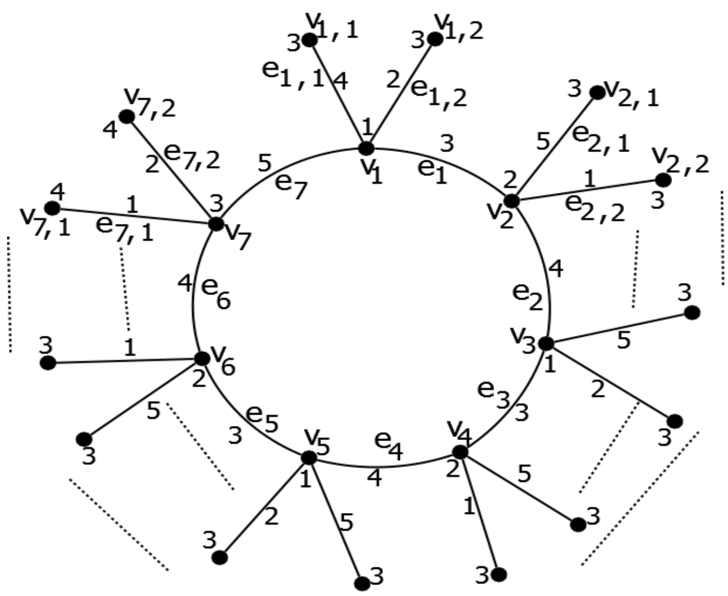

Figure 5: Total coloring of double crown graph C_7^{++}

References

[1]. M. Behzad, Graphs and their chromatic numbers, *Doctoral Thesis, Michigan State University,* (1965)

[2]. M. Behzad, Chartrand G and Cooper J.K., The color numbers of complete graphs, *Journal London Math. Soc.*, 42 (1967), 226-228

[3]. M., Behzad, A criterian for the planarity of the total graph of a graph, *Proc. Cambridge Philos. Soc.*, 63(1967), 679-681

[4]. O.V. Borodin, On the total coloring planar graphs, *J. Reine Angew Math.*, 394 (1989), 180-185

[5]. G. Jayaraman and D. Muthuramakrishnan, Total Chromatic Number of Double Star Graph Families, *Jour of Adv Research in Dynamical & control system.,* 10 (5)(2018), 631-635

[6]. A.V. Kostochka, the total coloring of a multigraph with maximal degree 4, *Discrete Math.*, 17(1989), 161-163.

[7]. S. Mohan, J. Geetha and Somasundaram K., Total coloring of Corona Product of two graphs, *Australasian Journal of Combinatorics.,* 68(1) (2017), 15-22.

[8]. M. Rosanfeld, On the total colouring of certain graphs, *Israel J. Math*., 9 (1972), 396-402.

[9]. N. Vijayaditya N, On total chromatic number of a graph, *J. London Math Soc*., (1971), 405- 408.

[10] V.G. Vizing, Some unsolved problems in graph theory, *Russian Mathematical Survey*., 23(6) (1968), 125-141.