

On the Edge Coloring of Triangular Snake Graph Families

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Received 2022 April 02; **Revised** 2022 May 20; **Accepted** 2022 June 18.

ABSTRACT

We discuss the edge chromatic number of the triangular snake graph T_n , double triangular snake graph DT_n , triple triangular snake graph TT_n and alternate triangular snake graph AT_n . A proper edge coloring of a graph G , is an assignment of colors to all the edges of graph G so that the adjacent edges received distinct colors. The smallest number of colors needed for such coloring is known as edge chromatic number.

Keywords: Triangular snake graph, double triangular snake graph, triple triangular snake graph, alternate triangular snake and edge coloring.

2010 Mathematics Subject Classification: 05C15.

1. INTRODUCTION

All graphs considered in this article are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph consists of a vertex set $V(G)$ and edge set $E(G)$ respectively. In 1880, Tait [4] was introduced the concept of edge coloring and he proved that, if the four-color conjecture is true then the edges of all the 3-connected planar graph can be colored properly only using 3-colors. In 1916, Konigsberg was proved that all the bipartite graphs have been edge colored with $\Delta(G)$

colors exactly. In 1949, Shannon[3] proved that all the graph have been edge colored with $\leq \frac{3}{2} \Delta(G)$ colors. In 1964,

Vizing[5] proved that for every simple graph G , $\chi'(G) \leq \Delta(G) + 1$.

An edge coloring of a graph G is that an assignment of colors to the edges of G such that the adjacent edges received distinct colors. The chromatic index of a graph G , denoted by $\chi'(G)$, is the minimum number of colors required for a proper edge coloring of graph G . The graph G is k -edge-chromatic if $\chi'(G) = k$. Obviously $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of a graph G .

In other words, An edge coloring of graph G is a function $c : E(G) \rightarrow \{1, 2, \dots, \Delta\}$, the colors satisfying the following conditions.

(i) $c(e) \neq c(e')$ for any two adjacent edges $e, e' \in E(G)$

The minimum number of colors are required for such coloring is called edge chromatic number of G and it is denoted by $\chi'(G)$.

Many real life situations can be modeled as a graph coloring problem, some of them are planning and scheduling problems, timetabling and map coloring. Since graph coloring problem is a NP-hard problem, until now there are not known deterministic methods as a whole that can solve such problems.

2. PRELIMINARIES

Definition 2.1: A *Triangular snake graph* [2] T_n is obtained from a path $\{u_1, u_2, \dots, u_n\}$ by joining u_k and u_{k+1} to a new vertex v_k for $k \in \{1, 2, \dots, n\}$. That is, every edge of a path is replaced by a triangle.

Definition 2.2: The *double triangular snake graph* [2] DT_n consists of two triangular snakes that have a common path.

Definition 2.3: The *triple triangular snake graph* [2] TT_n consists of three triangular snakes that have a common path.

Definition 2.4: An *alternate triangular snake graph* [2] AT_n is obtained by a path $\{u_1, u_2, \dots, u_n\}$ by joining u_k and u_{k+1} to a new vertex alternatively v_k for $k \in \{1, 3, 5, \dots\}$, i.e. Every alternate edge of a path is replaced by triangle.

In this paper, we focus on edge chromatic number for triangular snake graph T_n , double triangular snake graph DT_n , triple triangular snake graph TT_n and alternate triangular snake graph AT_n .

3. MAIN RESULTS

Theorem 3.1. Let T_n be the triangular snake graph of order $n \geq 3$, then $\chi'(T_n) = 4$.

Proof. Let $V(T_n) = \{u_l : 1 \leq l \leq n-1\} \cup \{v_l : 1 \leq l \leq n\}$ and

$E(T_n) = \{e_l : 1 \leq l \leq n-1\} \cup \{s_l : 1 \leq l \leq n-1\} \cup \{f_l : 1 \leq l \leq n-1\}$, where the edges $\{e_l : 1 \leq l \leq n-1\}$ represents the edge $\{v_l v_{l+1} : 1 \leq l \leq n-1\}$, the edges $\{s_l : 1 \leq l \leq n-1\}$ represents the edge $\{u_l v_l : 1 \leq l \leq n-1\}$ and the edges $\{f_l : 1 \leq l \leq n-1\}$ represents the edge $\{u_l v_{l+1} : 1 \leq l \leq n-1\}$

Define an edge coloring $c : E(T_n) \rightarrow \{1, 2, 3, \dots, \Delta\}$ as follows. Now we assign the edge coloring to all the edges as follows,

$$c(v_l v_{l+1}) = \begin{cases} 1, & \text{if } l \text{ is odd} \\ 2, & \text{if } l \text{ is even} \end{cases}$$

$$c(v_l u_l) = 3, c(u_l v_{l+1}) = 4$$

We observed that the procedure of edge coloring pattern, the graph T_n is edge colored properly with 4 colors. This implies that $\chi'(T_n) \leq 4$. Since $\Delta = 4$ and $\chi'(T_n) \geq \Delta = 4$. Therefore $\chi'(T_n) = 4$. Thus c is edge colored with 3 colors.

Theorem 3.2. Let DT_n be double triangular snake graph of order $n \geq 3$, then $\chi'(DT_n) = \Delta(DT_n) = 6$.

Proof. Let $V(DT_n) = \{u_l, w_l : 1 \leq l \leq n-1\} \cup \{v_l : 1 \leq l \leq n\}$ and

$$E(DT_n) = \left\{ \{e_l : 1 \leq l \leq n-1\} \cup \{e'_l : 1 \leq l \leq n-1\} \cup \{e''_l : 1 \leq l \leq n-1\} \cup \{s_l : 1 \leq l \leq n-1\} \cup \{s'_l : 1 \leq l \leq n-1\} \right\}, \text{ where the edges } \{e_l : 1 \leq l \leq n-1\}$$

represents the edge $\{v_l v_{l+1} : 1 \leq l \leq n-1\}$, the edges $\{e'_l : 1 \leq l \leq n-1\}$ represents the edge $\{u_l v_l : 1 \leq l \leq n-1\}$, the edges $\{e''_l : 1 \leq l \leq n-1\}$ represents the edge $\{u_l v_{l+1} : 1 \leq l \leq n-1\}$, the edges $\{s_l : 1 \leq l \leq n-1\}$ represents the edge $\{w_l v_l : 1 \leq l \leq n-1\}$ and the edges $\{s'_l : 1 \leq l \leq n-1\}$ represents the edge $\{w_l v_{l+1} : 1 \leq l \leq n-1\}$

Define an edge coloring $c : E(DT_n) \rightarrow \{1, 2, 3, \dots, \Delta\}$ as follows. Now we assign the edge coloring to all the edges as follows,

$$c(v_l v_{l+1}) = \begin{cases} 1, & \text{if } l \text{ is odd} \\ 2, & \text{if } l \text{ is even} \end{cases}$$

$$c(v_l u_l) = 3, \quad c(u_l v_{l+1}) = 4$$

$$c(v_l w_l) = 5, \quad c(w_l v_{l+1}) = 6$$

Clearly the above method of edge coloring, the graph DT_n is edge colored properly with 6 colors. This implies that $\chi'(DT_n) \leq 6$. Since $\Delta = 6$ and $\chi'(DT_n) \geq \Delta = 6$. Therefore $\chi'(DT_n) = 6$. Thus c is edge colored with 6 colors.

Theorem 3.3. Let TT_n be triple triangular snake graph, then $\chi'(TT_n) = \Delta(TT_n), n \geq 3$.

Proof. Let $V(TT_n) = \{u_l, s_l, w_l : 1 \leq l \leq n-1\} \cup \{v_l : 1 \leq l \leq n\}$ and

$$E(TT_n) = \left\{ \{e_l : 1 \leq l \leq n-1\} \cup \{e'_l : 1 \leq l \leq n-1\} \cup \{e''_l : 1 \leq l \leq n-1\} \cup \{e'''_l : 1 \leq l \leq n-1\} \cup \{f_l : 1 \leq l \leq n-1\} \cup \{f'_l : 1 \leq l \leq n-1\} \cup \{f''_l : 1 \leq l \leq n-1\} \right\}, \text{ where the edges } \{e_l : 1 \leq l \leq n-1\}$$

represents the edge $\{v_l v_{l+1} : 1 \leq l \leq n-1\}$, the edges $\{e'_l : 1 \leq l \leq n-1\}$ represents the edge $\{u_l v_l : 1 \leq l \leq n-1\}$, the edges $\{e''_l : 1 \leq l \leq n-1\}$ represents the edge $\{u_l v_{l+1} : 1 \leq l \leq n-1\}$, the edges $\{e'''_l : 1 \leq l \leq n-1\}$ represents the edge $\{u_l s_l : 1 \leq l \leq n-1\}$, the edges $\{f_l : 1 \leq l \leq n-1\}$ represents the edge $\{w_l v_l : 1 \leq l \leq n-1\}$, the edges $\{f'_l : 1 \leq l \leq n-1\}$ represents the edge $\{w_l v_{l+1} : 1 \leq l \leq n-1\}$ and the edges $\{f''_l : 1 \leq l \leq n-1\}$ represents the edge $\{s_l v_{l+1} : 1 \leq l \leq n-1\}$

Define an edge coloring $c : E(TT_n) \rightarrow \{1, 2, 3, \dots, \Delta\}$ as follows. Now we assign the edge coloring to all the edges as follows,

$$c(v_l v_{l+1}) = \begin{cases} 1, & \text{if } l \text{ is odd} \\ 2, & \text{if } l \text{ is even} \end{cases}$$

$$c(v_l u_l) = 3, \quad c(u_l v_{l+1}) = 4$$

$$c(v_l w_l) = 5, \quad c(w_l v_{l+1}) = 6$$

$$c(v_l s_l) = 7, \quad c(s_l v_{l+1}) = 8$$

We observed that the above condition of edge coloring, the graph TT_n is properly edge colored with 8 colors. Hence $\chi'(TT_n) \leq \Delta = 8$. Since $\Delta = 8$ and $\chi'(TT_n) \geq \Delta = 8$. Therefore $\chi'(TT_n) = 8$. Thus c is edge colored with 8 colors.

Theorem 3.4. Let AT_n be the alternate triangular snake graph, then $\chi'(AT_n) = \Delta(AT_n), n \geq 3$.

Proof. Let $V(AT_n) = \{u_l : l \in \{1, 2, \dots, n\}\} \cup \{v_l : l \in \{1, 3, 5, \dots, n-2\}\}$ and

$$E(TT_n) = \{e_l : l \in \{1, 2, \dots, n-1\}\} \cup \{e'_l : l \in \{1, 3, \dots, n-2\}\} \cup \{e''_l : l \in \{1, 3, \dots, n-2\}\},$$

where the edges $\{e_l : l \in \{1, 2, \dots, n\}\}$ represents the edge $\{u_l u_{l+1} : l \in \{1, 2, \dots, n-1\}\}$, the edges $\{e'_l : l \in \{1, 3, \dots, n-2\}\}$ represents the edge $\{u_l v_l : l \in \{1, 3, \dots, n-2\}\}$, the edges $\{e''_l : l \in \{1, 3, \dots, n-2\}\}$ represents the edge $\{v_l u_{l+1} : l \in \{1, 3, \dots, n-2\}\}$,

Define an edge coloring $c : E(AT_n) \rightarrow \{1, 2, 3\}$ as follows. Now we assign the edge coloring to all the edges as follows. Consider the following two cases

Case (i): when n is odd,

Subcase(i): $n = 2k + 1, k = 2, 4, 6, \dots$

$$c(e_l) = \begin{cases} 1, & \text{if } l \in \{1, 3, 5, \dots, n-2\} \\ 2, & \text{if } l = 2k-2, k \in \{2, 4, 6, \dots, n-3\} \\ 3, & \text{if } l = 2k+2, k \in \{1, 3, 5, \dots, n-1\} \end{cases}$$

For $k \in \{1, 3, 5, \dots, n-1\}$

$$c(e'_l) = \begin{cases} 2, & \text{if } l = 2k-1, \\ 3, & \text{if } l = 2k+2, \end{cases}$$

$$c(e''_l) = \begin{cases} 3, & \text{if } l = 2k-1, \\ 2, & \text{if } l = 2k+2, \end{cases}$$

Subcase(ii): $n = 2k + 3, k = 2, 4, 6, \dots$

$$c(e_l) = \begin{cases} 1, & \text{if } l \in \{1, 3, 5, \dots, n-2\} \\ 2, & \text{if } l = 2k-2, k \in \{2, 4, 6, \dots, n-1\} \\ 3, & \text{if } l = 2k+2, k \in \{1, 3, 5, \dots, n-3\} \end{cases}$$

For $k \in \{1, 3, 5, \dots, n-2\}$

$$c(e_l') = \begin{cases} 2, & \text{if } l = 2k-1, \\ 3, & \text{if } l = 2k+2, \end{cases}$$

$$c(e_l'') = \begin{cases} 3, & \text{if } l = 2k-1, \\ 2, & \text{if } l = 2k+2, \end{cases}$$

Case (i): when n is even,

$$c(e_l) = \begin{cases} 1, & \text{if } l \in \{1, 3, 5, \dots, n-1\} \\ 2, & \text{if } l = 2k-2, k \in \{2, 4, 6, \dots, n-4\} \\ 3, & \text{if } l = 2k+2, k \in \{1, 3, 5, \dots, n-2\} \end{cases}$$

For $k \in \{1, 3, 5, \dots, n-1\}$

$$c(e_l') = \begin{cases} 2, & \text{if } l = 2k-1, \\ 3, & \text{if } l = 2k+2, \end{cases}$$

$$c(e_l'') = \begin{cases} 3, & \text{if } l = 2k-1, \\ 2, & \text{if } l = 2k+2, \end{cases}$$

We have observed that the above condition of edge coloring, the graph AT_n is properly edge colored with 3 colors. This implies that $\chi'(T_n) \leq \Delta = 3$. Since $\Delta = 3$ and $\chi'(T_n) \geq \Delta = 3$. Therefore $\chi'(T_n) = 3$. Thus c is edge colored with 3 colors.

4. CONCLUSION

In this article, we obtained an edge chromatic number of triangular snake graph T_n , double triangular snake graph DT_n , triple triangular snake graph TT_n and alternate triangular snake graph AT_n .

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