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Energy Bounds of Soft Graphs

P. Sakthi Amaravathi [a], K. Rajendran [b]

[a] *Research Scholar, Department of Mathematics, Vels Institute of Science, Technology & Advanced Studies, Chennai, Tamil Nadu, India. E-mail: p.sakthi3@gmail.com, Orcid: https://orcid.org/0009-0000-3841-9606*
[b] *Associate Professor, Department of Mathematics, Vels Institute of Science, Technology & Advanced Studies, Chennai, Tamil Nadu, India. E-mail: gkrajendra59@gmail.com, Orcid: https://orcid.org/0000-0003-1682-5421*

Abstract

The paper examines energy in soft graphs and establishes the maximum energy such graphs can contain. It introduces several key concepts, namely equi-energy soft graphs, hyper-energy soft graphs, non-hyper-energy soft graphs, and hypo-energy soft graphs. These amounts of energy less than the number of their vertices are called hypo-energy soft graphs, and their counterparts, which surpass complete graph energy, are called hyper-energy; these categories display structural properties under uncertainty, and find use in the analysis of networks. These concepts facilitate a deeper understanding of energy within the framework of soft graph theory. The paper also investigates the maximum and minimum energy levels achievable by soft graphs, taking into account variables such as the number of vertices and edges. Furthermore, it tackles the challenges associated with computing eigenvalues for the adjacency matrices of soft graphs. The paper substantiates these concepts with diverse examples that demonstrate their application and importance. These findings extend traditional graph energy theory to address uncertain scenarios and open new avenues for research in soft computing and network analysis.

Keywords: Soft Graph, Energy, Equienergetic, Hyper-Energetic, Hypo-Energetic.

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Introduction

Soft graph theory, an extension of classical graph theory, provides a flexible framework for modelling uncertain or vague relationships between objects. Energy, a fundamental concept in graph theory, has been widely studied for its applications in various fields. Soft set theory, introduced by Molodtsov (Onyeozili & Gwary, 2014) and further developed in (Balakrishnan, 2004; DiStefano & Davis, 2009), provides a flexible framework for handling uncertainty. Graph energy concepts (Thenge et al., 2020; Akram & Nawaz, 2015) and matrix-based approaches have enriched graph theory. Recent studies on soft graphs and their energy (Sakthi Amaravathi & Rajendran, 2025) integrate these ideas, extending structural analysis under uncertain environments.

In this study, we proposed the concept of energy in soft graphs, exploring bounds on the energy of soft graphs (ESG). We define the concepts of equienergetic soft graphs with illustrations, hyper-energetic soft graphs, and hypo-energetic soft graphs, providing new insights into energy properties in soft graph theory.

Preliminaries

Definition 1. Consider U be a universal set and $A \subseteq U$ be a parameter set. Then (F, A) is called a soft set over the universal set U, where F is mapping from A to (U).

Definition 2. Let $G^* = (V, E)$ be a simple connected graph and $\mathbb{A} \subseteq U$ be a parameter set. Then $(S,\ \mathbb{A})$ is a soft set over V, $(T,\ \mathbb{A})$ is also a soft set over E and the 4-tuple $G = (G^*, \text{Ş}, \text{Ţ}, \mathbb{A})$ is defined as a soft graph which is represented by $G = (G^*, \text{Ş}, \text{Ţ}, \mathbb{A}) = \ \{(x) = (((x), (x)),\ for\ all\ x \epsilon \mathbb{A}\}$

Definition 3. Consider $\mathcal{A}_{SG}\,(F, \mathbb{A})$ be the adjacency matrix of soft graph G, and assume G = (G*,Ş , Ţ, 𝔸) be a soft graph of a simple connected graph G*. The sum of the absolute values of a soft graph G's eigenvalues is then used to define its energy.

That is, $\mathcal{E}_{SG}\,(G) = \sum_{i=1}^{n} |\lambda_i|$.

Note that, Energy of soft graph (ESG) is represented as $\mathcal{E}_{SG}\,(G)$.

Bounds on the Energy of Soft Graphs

As such there is no general method to define the characteristic polynomial of a soft graph and therefore to determine a soft graph's eigenvalues is still more difficult; and hence to calculate the ESG is difficult task, however the researchers were able to find some upper and lower bounds for $\mathcal{E}_{SG}$ in the terms of soft graph theoretic parameters such as the number of edges etc.

Theorem 1

Let D be the absolute value of the determinant of adjacency matrix A of soft Graph G.

Then $\sqrt{2p + q(q-1)D^{2/q}} \le \mathcal{E}_{SG} \le \sqrt{2pq}$, where q is the vertices and p is the edges of the soft graph G.

Proof

We start with an identity for $\mathcal{E}_{SG}$,

that is $\mathcal{E}_{SG}(G) = \sum_{i=1}^{q} |\lambda_i|$

$$\mathcal{E}_{SG}^2(G) = \left(\sum_{i=1}^{q} |\lambda_i|\right)^2$$

$$= \sum_{i=1}^{q} \lambda_i^2 + \sum_{i \neq j} |\lambda_i||\lambda_j|$$

$$= 2p + \sum_{i \neq j} |\lambda_i||\lambda_j|$$

$$= 2p + q(q-1)AM(|\lambda_i \lambda_j|)$$

$$\ge 2p + q(q-1)GM(|\lambda_i \lambda_j|)$$

Now, $GM(|\lambda_i \lambda_j|) = \left(\prod_{i<j} |\lambda_i \lambda_j|\right)^{2/q^2-q}$

$$= \left(\prod_i |\lambda_i|^{n-1}\right)^{2/q^2-q}$$

$$= \left(\prod_i |\lambda_i|\right)^{2/q}$$

$= |\det A|^{2/q}$

$$\mathrm{GM}(|\lambda_i\lambda_j|) = D^{2/q}$$

$\therefore \mathcal{E}_{SG}^2(G) \geq 2p + q(q-1)D^{2/q}$

To prove the upper bound

Consider the identity,

$q\sum_{i=1}^{q}\lambda_i^2 - \left(\sum_{i=1}^{q}|\lambda_i|\right)^2 = \sum_{i\neq j}^{q}\left(|\lambda_i| - |\lambda_j|\right)^2$

Which is a special case of Lagrange's identity, we can get

$$2pq - \mathcal{E}_{SG}^2(G) = \sum_{i\neq j}^{q}\left(|\lambda_i| - |\lambda_j|\right)^2 \geq 0$$

$$\mathcal{E}_{SG}^2(G) \leq 2pq$$

$$\sqrt{2p + q(q-1)D^{2/q}} \leq \mathcal{E}_{SG}(G) \leq \sqrt{2pq}$$

Theorem 2

The energy of soft graph $\mathcal{E}_{SG}(G)$ of any soft graph with q vertices and p edges satisfies the following condition: $\mathcal{E}_{SG}(G) = \frac{2p}{q} + \sqrt{(q-1)\left\{2p - \left(\frac{2p}{q}\right)^2\right\}}$.

Proof

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of soft graph G and λ_n be the maximum eigenvalue of soft graph G. Then the Cauchy-Schwarz inequality, if $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$ are the positive k-vectors, then

$$\left(\sum_{i=1}^{k} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{k} x_i^2\right)\left(\sum_{i=1}^{k} y_i^2\right)$$

Let $x_i = 1, y_i = |\lambda_i|$ and $k = n-1$, then

$$\left(\sum_{i=1}^{n-1}|\lambda_i|\right)^2 \leq (n-1)\left(\sum_{i=1}^{n-1}|\lambda_i|^2\right) \qquad --(1)$$

We have

$$\mathcal{E}_{SG}(G) = \sum_{i=1}^{n}|\lambda_i| = \sum_{i=1}^{n-1}|\lambda_i| + |\lambda_n|$$

Therefore

$$\mathcal{E}_{SG}(G) - |\lambda_n| = \sum_{i=1}^{n-1}|\lambda_i| \qquad --(2)$$

And

$$2p = \sum_{i=1}^{n}|\lambda_i|^2 = \sum_{i=1}^{n-1}|\lambda_i|^2 + |\lambda_n|^2$$

Therefore

$$2p - |\lambda_n|^2 = \sum_{i=1}^{n-1}|\lambda_i|^2 \qquad --(3)$$

Now the expression (1) becomes,

$$(\mathcal{E}_{SG}^2(G) - |\lambda_n|^2) \leq (n-1)(2p - |\lambda_n|^2)$$

Take $\lambda_n = r \geq 0$ and $2p = nr$, then we have,

$$(\mathcal{E}_{SG}^2(G) - r^2) \leq (n-1)(2nr - r^2)$$

Therefore

$$\mathcal{E}_{SG}^2(G) = r^2 + (n-1)(2nr - r^2)$$

$$\mathcal{E}_{SG}(G) \leq r + \sqrt{(n-1)(2nr - r^2)}$$

$$\mathcal{E}_{SG}(G) = \frac{2p}{q} + \sqrt{(q-1)\left\{2p - \left(\frac{2p}{q}\right)^2\right\}}$$

The proof is completed.

Example 1

Consider a simple graph G^* as shown in fig. 1.

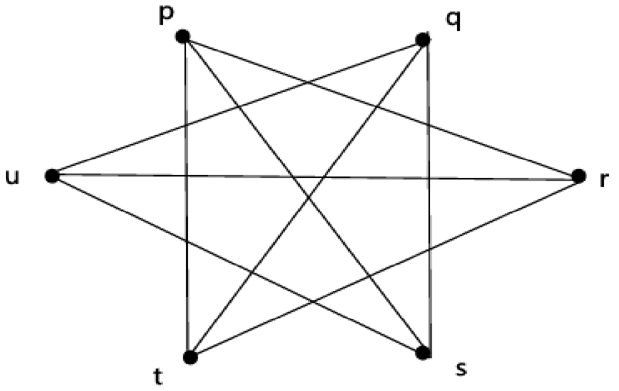

Fig. 1: Simple Graph $G*$

Let $\mathbb{B} = \{r, s\}$ be a parameter set.

We defined an approximate function $S_1: \mathbb{B} \to P(V)$ by

$S_1(x) = \{y \in V / xRy \Leftrightarrow d(x, z) \leq 1\}$ for all $x \in \mathbb{B}$.

That is, $S_1(r) = \{p, r, t\}$, $S_1(s) = \{q, s, u\}$.

We defined an approximate function $T_1: \mathbb{B} \to P(E)$ by

$T_1(x) = \{uv \in E / \{u, v\} \subseteq S_1(x)\}$ for all $x \in \mathbb{B}$.

That is, $T_1(r) = \{pr, rt, tp\}$, $T_1(s) = \{qs, su, uq\}$.

Thus, $H_1(r) = \left(S_1(r), T_1(r)\right), H_1(s) = \left(S_1(s), T_1(s)\right)$ are subgraphs of G^*.

$\therefore G = \{H_1(a), H_1(c)\}$ is a soft graph of G^*, $G = G^*$

Then

$$\mathcal{A}_{SG}(G) = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The eigenvalues of G is – 2, - 1, - 1, 1, 1, 2. $\mathcal{E}_{SG}(G) = 8$

Here, the number of vertices q =6, the number of edges p = 9, det (G) = 0.

We know that, $\sqrt{2p + q(q-1)D^{2/q}} \leq \mathcal{E}_{SG}(G) \leq \sqrt{2pq}$ -------------------------(A)

Lower bound of $\mathcal{E}_{SG}(G)$

$$\sqrt{2p + q(q-1)D^{2/q}} = \sqrt{2(9) + 6(6-1)D^{2/6}} = 4.242$$

Upper bound of $\mathcal{E}_{SG}(G)$

$$\sqrt{2pq} = 76.36$$

Degree based $\mathcal{E}_{SG}(G)$

$$\mathcal{E}_{SG}(G) = \frac{2p}{q} + \sqrt{(q-1)\left\{2p - \left(\frac{2p}{q}\right)^2\right\}} = 9.708$$

Types on Energy of Soft Graph

Definition 4. Two non - isomorphic soft graphs $G_1$and G_2 of same order are said to be equienergetic soft graph if $\mathcal{E}_{SG}(G_1) = \mathcal{E}_{SG}(G_2)$.

Definition 5. A soft graph of energy is greater than their complete soft graph of energy which soft graph is called Hyper-energetic soft graph.

Definition 6. If $n < 8$ vertices of a soft graph, then the soft graph is called non-hyper-energetic soft graph.

Remark 1. A hyper-energetic soft graph with n vertices exists for all $n \geq 8$ and there is no hyper-energetic soft graph.

Definition 7. A soft graph on order n is said to be hypo-energetic if E(G) is less than its order otherwise it is said to be non-hypo-energetic soft graph.

Example 2

Let $\mathcal{A} = \{a, c, e\}$ and $\mathcal{B} = \{p, q, r, s, t, u\}$ be the parameterized sets of G_1^* and G_2^*.

Then the soft graph $G_1$and G_2 are depicted in fig. 2.

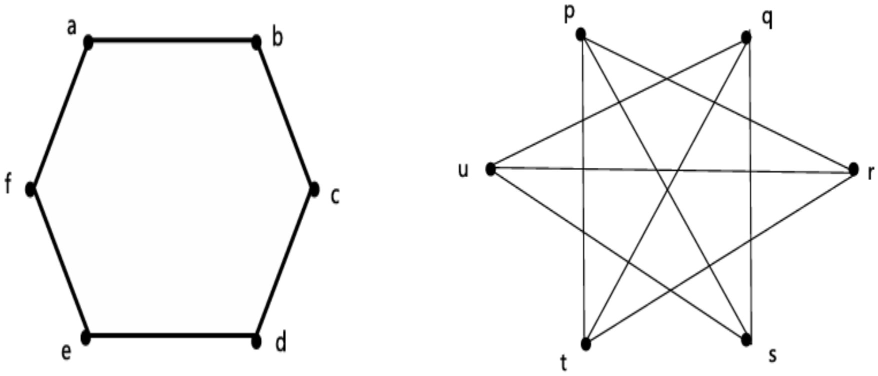

Fig. 2: Soft Graph $G1$and $G2$

Let $\mathbb{A} = \{a, c, e\}$ be a parameter set.

An approximate function is given as $S_1: \mathbb{A} \rightarrow P(V_1^*)$ by

$S_1(x) = \{y \in V_1^*/xRy \Leftrightarrow d(x, z) \leq 1\}$ for all $x \in \mathbb{A}$.

That is, $S_1(b) = \{a, b, f\}$, $S_1(b) = \{b, c, d\}$ and $S_1(f) = \{d, e, f\}$.

An approximate function is given as $T_1: \mathbb{A} \rightarrow P(E)$ by

$T_1(x) = \{uv \in E/\{u, v\} \subseteq S_1(x)\}$ for all $x \in \mathbb{A}$.

That is, $T_1(a) = \{ab, af\}$, $T_1(c) = \{bc, cd\}, T_1(e) = \{de, ef\}$.

Thus, $H_1(a) = (S_1(a), T_1(a)), H_1(c) = (S_1(c), T_1(c)), H_1(e) = (S_1(e), T_1(e))$ are subgraphs of G_1^*.

$\therefore G_1 = \{H_1(a), H_1(c), H_1(e)\}$ is a soft graph of G^*, $G_1 = G_1^*$.

From the example 2.4, G_2 is a soft graph of G_2^*, $G_2 = G_2^*$.

Here $G_1$and G_2 are non- isomorphic soft graphs with the same number of vertices.

The energy of the soft graph $G_1$is $\mathcal{E}_{SG}(G_1) = 8$

The energy of the soft graph $G_2$is $\mathcal{E}_{SG}(G_2) = 8$

∴ These soft graphs are called equienergitic soft graph.

Example 3

Let $\mathcal{A} = \{a, b, c, d, e\} \subseteq V_1(G_1^*)$ and $\mathcal{B} = \{k, l\} \subseteq V_2(G_2^*)$ be the parameter sets.

Then the soft graph G_1 is shown in fig. 3 and complete soft graph G_2 is also shown in the fig. 4.

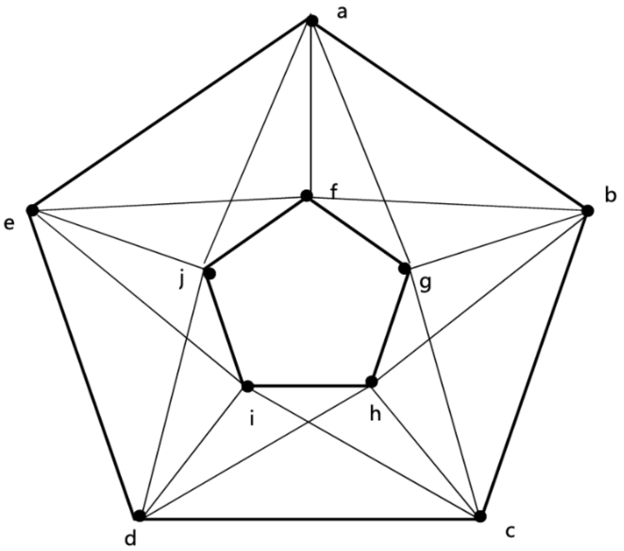

Fig. 3: $G1$ Petersen Graph

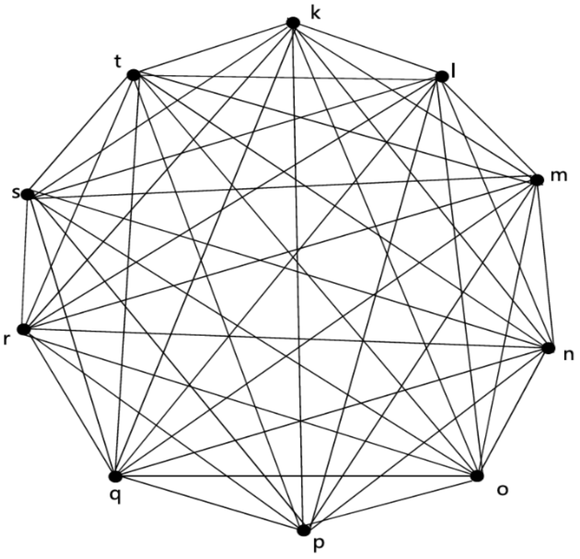

Fig. 4: Complete Soft Graph $G2$

The energy of the soft graph $G_1$is $\mathcal{E}_{SG}(G_1) = 18.9444$

The energy of the soft graph $G_2$is $\mathcal{E}_{SG}(G_2) = 18$

Here, the energy of soft graph G_1 is greater than their energy of complete soft graph.

∴ These soft graphs are called hyper-energetic soft graph.

Example 4

Let $\mathcal{A} = \{w\} \subseteq V(G^*)$ is a parameter set.

Then the soft graph G is illustrated in fig. 5.

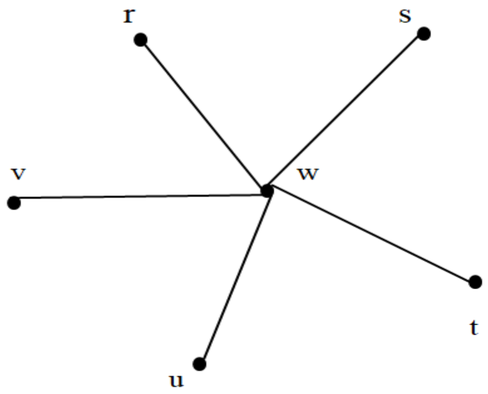

Fig. 5: Soft Graph G

The ESG of G is $\mathcal{E}_{SG}(G)$ = 4.47

The ESG of G is less than the number of vertices.

∴ The soft graph is a hypo-energetic soft graph.

Example 5

Let a simple connected graph G^*

Let $\mathbb{A} = \{a, c, e\}$ be a parameter set.

An approximate function be defined as $S_1: \mathbb{A} \to P(V)$ by $S_1(x) = \{y \in V/xRy \Leftrightarrow d(x,z) \leq 1\}$ for all $x \in \mathbb{A}$. That is, $S_1(b) = \{a, b, f\}$, $S_1(b) = \{b, c, d\}$ and $S_1(f) = \{d, e, f\}$.

An approximate function be defined as $T_1: \mathbb{A} \to P(E)$ by $T_1(x) = \{uv \in E/\{u, v\} \subseteq S_1(x)\}$ for all $x \in \mathbb{A}$. That is, $T_1(a) = \{ab, af\}$, $T_1(c) = \{bc, cd\}, T_1(e) = \{de, ef\}$.

Thus, $H_1(a) = (S_1(a), T_1(a)), H_1(c) = (S_1(c), T_1(c)), H_1(e) = (S_1(e), T_1(e))$ are subgraphs of G^*.

∴ $G = \{H_1(a), H_1(c), H_1(e)\}$ is a soft graph of G^*.

Here $C = \cup_{x \in C} S_1(x) = \{a, b, c, d, e, f\}$

$$\mathcal{A}_{SG}(G) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues of G are $\lambda = -3, 0, 0,0, 0, 3$

Energy of soft graph G is $\mathcal{E}_{SG}(G)$ = 6.

Here n = 6, $\mathcal{E}_{SG} = 6$

∴ G is a non-hypo - energitic soft graph.

Example 6

Consider $G^* = P_5$ and $V(G^*) = \{r, s, t, u, v\}$

Let $\mathbb{A} = \{s, t\}$ be a parameter set.

An approximate function be defined as $S_1: \mathbb{A} \to P(V)$ by $S_1(x) = \{y \in V/xRy \Leftrightarrow d(x,z) \leq 1\}$ for all $x \in \mathbb{A}$. That is,

$S_1(s) = \{r, s, t\}, S_1(t) = \{s, t, u\}$.

An approximate function be defined as $T_1: \mathbb{A} \rightarrow P(E)$ by $T_1(x) = \{uv \in E/\{u, v\} \subseteq S_1(x)\}$ for all $x \in \mathbb{A}$. That is, $T_1(s) = \{rs, st\}$, $T_1(t) = \{st, tu\}$.

Thus, $H_1(s) = \left(S_1(s), T_1(s)\right), H_1(t) = \left(S_1(t), T_1(t)\right)$ are subgraphs of G^*.

$\therefore G = \{H_1(s), H_1(t)\}$ is a soft graph of G^*.

Here $C = \bigcup_{x \in C} S_1(x) = \{r, s, t, u, v\}$

$$\mathcal{A}_{SG}(G) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues of G are $\lambda = -1.6180, -0.6180, o.6180, 1.6180$

Energy of soft graph G is $\mathcal{E}_{SG}(G) = 4.472$.

Here n = 4, $\mathcal{E}_{SG} = 4.472$

∴ G is a non-hypo - energitic soft graph.

Conclusion

In this article, we have proposed the notion of energy in soft graphs, establishing bounds on the ESG. We defined and illustrated with examples the notions of equienergetic soft graphs, hyper-energetic soft graphs, and hypo-energetic soft graphs, providing a deeper understanding of energy properties in soft graph theory. This work contributes to the development of soft graph theory, offering new avenues for research and applications in uncertain or vague environments.

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