Advances and Applications in Discrete Mathematics
© 2021 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/DM027010031
Volume 27, Number 1, 2021, Pages 31-38 **P-ISSN: 0974-1658**

TOTAL COLORING OF CERTAIN GRAPHS

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Abstract

A total coloring of a graph *G* is an assignment of colors to both vertices and edges of *G*, such that no two adjacent or incident vertices of *G* receive the same color. In this paper, we have discussed the total coloring and total chromatic number of twig graph T_n, splitting graph of comb graph $S'(P_n^+)$, and shadow graph of comb graph $D_2(P_n^+)$.

1. Introduction

All graphs considered here are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$, respectively. A *coloring* of a graph *G* is an assignment of colors to the vertices or edges or both. A *total coloring* of *G* is a mapping $f : V(G) \cup E(G) \to C$, where C is the set of colors satisfying the following three conditions:

Received: September 24, 2020; Revised: December 5, 2020; Accepted: February 20, 2021
2020 Mathematics Subject Classification: 05C15.
Keywords and phrases: twig graph, comb graph, splitting graph, shadow graph, total coloring, total chromatic number.

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(i) No two adjacent vertices receive the same color.

(ii) No two adjacent edges receive the same color.

(iii) No edges and its end vertices receive the same color.

The *total chromatic number* of a graph *G*, denoted by $\chi''(G)$, is the minimum number of colors that suffice in a total coloring. It is clear that $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of *G*. The concept of total coloring was introduced by Behzad [1] and Vizing [8]. Also, they have posed the conjecture that for any simple graph *G*, $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$. If a graph *G* is total colorable with $\Delta(G) + 1$ colors, then the graph is called of type-I, and if it is total colorable with $\Delta(G) + 2$ colors, then it is of type-II. Rosenfeld [5] and Vijayaditya [7] verified the "total coloring conjecture (TCC)", for any graph *G* with maximum degree ≤ 3. Borodin [2] verified the TCC for maximum degree ≥ 9 in planar graphs. Muthuramakrishnan and Jayaraman [4] obtained the total chromatic number of splitting graph of a path, a cycle and a star graph. In the present work, we investigate the total chromatic number of a twig graph, splitting graph of a comb graph, and shadow graph of a comb graph.

2. Preliminaries

Definition 2.1 [6]**.** The *splitting graph* of a graph *G* is obtained by adding to each vertex *v*, a new vertex v' such that v' is adjacent to every vertex that is adjacent to *v* in *G*, which means $N(v) = N(v')$. It is denoted by $S'(G)$.

Definition 2.2. The *shadow graph* $D_2(G)$ of a graph *G* is constructed by taking two copies of *G* say G' and G'' and joining each vertex u' in G' to the neighbours of the corresponding vertex u'' in G''.

Definition 2.3 [3]**.** The *comb graph* P_n^+ is the graph obtained from a path by attaching pendent edge at each vertex of a path.

Definition 2.4 [3]**.** A *twig graph* is a tree obtained from a path by attaching exactly two pendant edges to each internal vertex of the path and is denoted by T_n.

3. Results and Discussion

Theorem 3.1. *Let* T_n *be the twig graph. Then its total chromatic number is* $\Delta(T_n) + 1$.

Proof. Let

$$V(T_n) = \{u_k, w_k : 1 \le k \le n-2\} \cup \{v_k : 1 \le k \le n\}$$

and

$$E(T_n) = \{u_k v_{k+1}, w_k v_{k+1} : 1 \le k \le n-2\} \cup \{v_k v_{k+1} : 1 \le k \le n-1\}.$$

We construct the total coloring $f : S \to C$, where $S = V(T_n) \cup E(T_n)$ and the set of colors $C = \{1, 2, 3, 4, 5\}$. Now, we apply the total coloring to these vertices and edges as follows:

For $1 \le k \le n$,

$$f(v_k) = \begin{cases} 1, & \text{if } k \text{ is odd}, \\ 2, & \text{if } k \text{ is even}. \end{cases}$$

For $1 \le k \le n-2$,

$$f(u_k) = 3, \quad f(w_k) = 4, \quad f(u_k v_{k+1}) = 5,$$

$$f(v_{k+1} w_k) = \begin{cases} 1, & \text{if } k \text{ is odd}, \\ 2, & \text{if } k \text{ is even}. \end{cases}$$

For $1 \le k \le n-1$,

$$f(v_k v_{k+1}) = \begin{cases} 3, & \text{if } k \text{ is odd}, \\ 4, & \text{if } k \text{ is even}. \end{cases}$$

By using the above rule of total coloring, the graph T_n is properly total colored with $\Delta(T_n) + 1$ colors. Hence, for the total chromatic number of the twig graph T_n, $\chi''(T_n) = \Delta(T_n) + 1$.

Illustration 3.2. Total coloring of twig T_6 as shown in Figure 1.

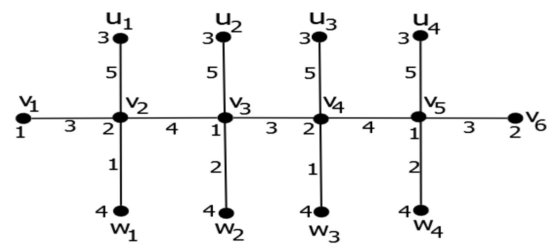

Figure 1

Theorem 3.3. *Let* $S'(P_n^+)$ *be the splitting graph of a comb graph. Then its total chromatic number is* $\Delta(S'(P_n^+)) + 1$.

Proof. Let

$$V(P_n^+) = \{v_k, w_k : 1 \leq k \leq n\}$$

and

$$E(P_n^+) = \{u_k w_k, v_k v_{k+1} : 1 \leq k \leq n\}.$$

Now, construct the splitting graph of a comb graph by adding the new vertices $\{u_k : 1 \leq k \leq n\}$ corresponding to the vertices $\{v_k : 1 \leq k \leq n\}$ and $\{s_k : 1 \leq k \leq n\}$ corresponding to the vertices $\{w_k : 1 \leq k \leq n\}$ of P_n^+ and joining them to obtain $S'(P_n^+)$. Thus, the vertex set and the edge set of splitting graph of a comb graph are as given below:

$$S'(V(P_n^+)) = \{u_k, v_k, w_k, s_k : 1 \leq k \leq n\},$$

$$S'(E(P_n^+)) = \{v_k w_k, u_k w_k, v_k s_k : 1 \leq k \leq n\}$$

$$\cup \{v_k v_{k+1}, v_k u_{k+1}, u_k v_{k+1} : 1 \leq k \leq n-1\}.$$

Define a total coloring $f: S \to C$, where $S = V(S'(P_n^+)) \cup E(S'(P_n^+))$ and the set of colors $C = \{1, 2, 3, 4, 5, 6, 7\}$. Assign the total coloring to these vertices and edges as follows:

For $1 \leq k \leq n$,

$$f(u_k) = 4, \quad f(s_k) = 3, \quad f(w_k) = 5,$$

$$f(v_k) = \begin{cases} 1, & \text{if } k \text{ is odd}, \\ 2, & \text{if } k \text{ is even}, \end{cases}$$

$$f(u_k w_k) = 6, \quad f(v_k s_k) = 6,$$

$$f(v_k w_k) = \begin{cases} 2, & \text{if } k \text{ is odd}, \\ 1, & \text{if } k \text{ is even}. \end{cases}$$

For $1 \leq k \leq n-1$,

$$f(v_k v_{k+1}) = \begin{cases} 3, & \text{if } k \text{ is odd}, \\ 4, & \text{if } k \text{ is even}, \end{cases}$$

$$f(u_k v_{k+1}) = 7, \quad f(v_k u_{k+1}) = 5.$$

It is clear that the graph $S'(P_n^+)$ is properly total colored with $\Delta(S'(P_n^+)) + 1$ colors. Hence, the total chromatic number of the splitting graph of the comb graph is given by $\chi''(S'(P_n^+)) = \Delta(S'(P_n^+)) + 1$.

Illustration 3.4. Total coloring of splitting graph of a comb graph $S'(P_6^+)$ as shown in Figure 2.

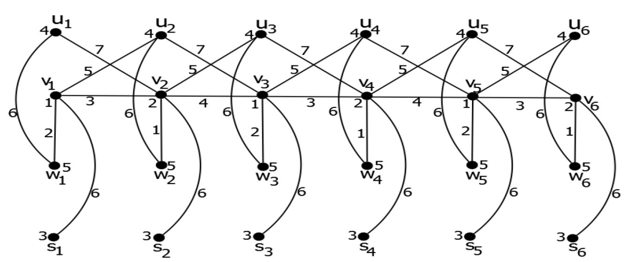

Figure 2

Theorem 3.5. *Let* $D_2(P_n^+)$ *be the shadow graph of the comb graph. Then its total chromatic number is* $\Delta(D_2(P_n^+))+1$.

Proof. Let

$$V(P_n^+) = \{u_k, v_k : 1 \le k \le n\}$$

and

$$E(P_n^+) = \{u_k v_k : 1 \le k \le n\} \cup \{v_k v_{k+1} : 1 \le k \le n-1\}.$$

We construct the shadow graph of the comb graph. Take two copies of the comb graph, say $(P_n^+)'$ and $(P_n^+)''$. Now, join each vertex of $(P_n^+)'$ to the neighbours of the corresponding vertex of $(P_n^+)''$. Thus, the vertex set and the edge set of the shadow graph of the comb graph are as given below:

$$V(D_2(P_n^+)) = \{u_k, v_k, s_k, w_k : 1 \le k \le n\}$$

and

$$E(D_2(P_n^+)) = \{u_k v_k, u_k w_k, s_k v_k, s_k w_k : 1 \le k \le n\}$$
$$\cup \{v_k v_{k+1}, w_k w_{k+1}, v_k w_{k+1}, w_k v_{k+1} : 1 \le k \le n-1\}.$$

Define a total coloring $f : S \to C$, where $S = V(D_2(P_n^+)) \cup E(D_2(P_n^+))$ and the set of colors $C = \{1, 2, 3, 4, 5, 6, 7\}$.

Assign the total coloring to these vertices and edges as follows:

For $1 \le k \le n$,

$$f(u_k) = 4, \quad f(s_k) = 4,$$

$$f(v_k) = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ 2, & \text{if } k \text{ is even,} \end{cases}$$

$$f(w_k) = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ 2, & \text{if } k \text{ is even,} \end{cases}$$

$$f(u_k w_k) = 6, \quad f(v_k s_k) = 6,$$

$$f(u_k v_k) = \begin{cases} 2, & \text{if } k \text{ is odd,} \\ 1, & \text{if } k \text{ is even,} \end{cases}$$

$$f(s_k w_k) = \begin{cases} 2, & \text{if } k \text{ is odd,} \\ 1, & \text{if } k \text{ is even.} \end{cases}$$

For $1 \leq k \leq n-1$,

$$f(v_k v_{k+1}) = \begin{cases} 3, & \text{if } k \text{ is odd,} \\ 4, & \text{if } k \text{ is even,} \end{cases}$$

$$f(w_k w_{k+1}) = \begin{cases} 3, & \text{if } k \text{ is odd,} \\ 4, & \text{if } k \text{ is even,} \end{cases}$$

$$f(w_k v_{k+1}) = 5, \quad f(v_k w_{k+1}) = 7.$$

By using the above pattern of the total coloring, the graph $D_2(P_n^+)$ is properly total colored with $\Delta(D_2(P_n^+)) + 1$ colors. Hence, the total chromatic number of the shadow graph is given by $\chi''(D_2(P_n^+)) = \Delta(D_2(P_n^+)) + 1$.

Illustration 3.6. Total coloring of the shadow graph of the comb graph $D_2(P_6^+)$ as shown in Figure 3.

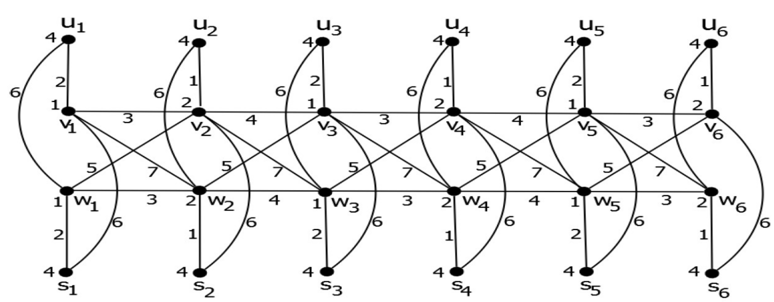

Figure 3

Acknowledgement

The authors thank the anonymous referees for their valuable suggestions towards the improvement of the manuscript.

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