

**EQUITABLE EDGE COLORING OF SUBDIVISION FOR
SPLITTING AND SHADOW GRAPHS**

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Abstract

An equitable edge coloring of a graph G is a proper edge coloring of G where the sizes of any two color classes differ by at most one. In this paper, we determine the Equitable Edge Chromatic Number (EECN) for $S(S'(P_n)), S(S'(C_n)), S(S'(K_{1,n})), S(D_2(P_n)), S(D_2(C_n))$ and $S(D_2(K_{1,n}))$ respectively.

Keywords: Subdivision graph, splitting graph, shadow graph, equitable edge coloring, equitable edge chromatic number.

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1. Introduction

Let G be a graph with $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G , respectively. In 1880, the fundamental article of edge coloring was written by Tait. A p -edge coloring of G is an edge coloring that uses exactly p distinct colors, say c_1, c_2, \dots, c_p . Let $E(c_i)$ represent the set of edges assigned the color c_i for $i = 1, 2, \dots, p$ and $d(v)$ -degree of a vertex v . The chromatic index $\chi'(G)$, is the smallest number of colors needed to color the edges of G . Clearly $\chi'(G) \geq \Delta(G)$.

In 1964, Vizing [9] proved that the chromatic index $\chi'(G)$ satisfies $\chi'(G) \leq \Delta(G) + 1$. Later, Meyer[7] established the concept of equitable coloring and equitable chromatic number in 1973. The notion of equitable edge coloring was established by Hilton and de Werra[2]. It involves assigning colors to all the edges of a graph G such that the edges are partitioned into color classes, with the size difference between any two classes being at most one.

An edge coloring of G is said to be equitable, if the number of edges in any two color classes differ by atmost one. An equitable edge chromatic number $\chi'_e(G)$ is minimum number of colors suffice to color the edges of the graph. A color class is a set of edges that are collect the same color. Kumar and Jayaraman [5] calculated the EEC for prism, anti-prism and n-crossed prism graph. Jagannathan et al.[3] determined the EECN for splitting graph of W_n, DW_n and G_n . Veninstine Vivik et al.[8] proved the EECN for splitting graph of helm and sunlet graph. In 2020, Manikandan et al.[6] proved that EEC of strong product of path and cycle.

2. Preliminaries

Definition 2.1. For a simple graph G , let f be a proper k - edge coloring of G , such that $\|E_i| - |E_j|\| \leq 1, i, j = 1, 2, \dots, k, \chi'_e(G) = \min \{k/ k\text{- equitable edge coloring of } G\}$

Definition 2.2. For a graph G , the splitting graph $S'(G)$ [4] of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.

Definition 2.3. The shadow graph $D_2(G)$ is constructed by taking two copies of G say G' and G'' . Join each vertex u' in G' to the neighbors of the corresponding vertex v' in G'' .

Definition 2.4. The subdivision graph $S(G)$ of the graph G is obtained from G by adding a vertex in the middle of every edge of G

Lemma 2.5 [1]. For any simple graph $G, \chi'_e(G) \geq \Delta(G)$.

Lemma 2.6[1]. For any simple graph G and $H, \chi'_e(G) \geq \chi'(G)$ and if $H \subseteq G$ then $\chi'(H) \leq \chi'(G)$, where $\chi'(G)$ is the proper edge chromatic number of G .

3. Results and Discussions

Theorem 3.1: Let $S(S'(P_n))$ be the subdivision of splitting graph of path then $\chi'_e(S(S'(P_n))) = 4, n \geq 3$.

Proof: Let $S'(P_n)$ contains $2n$ vertices and $3n - 3$ edges and so its subdivision graph $S(S'(P_n))$ has following vertices and edges as follows

Let $V(S(S'(P_n))) = \{u_i, v_i : 1 \leq i \leq n\} \cup \{u'_i, v'_i, v''_i : 1 \leq i \leq n - 1\}$ and

$E(S(S'(P_n))) = \{e_i, e'_i, f_i, f'_i, g_i, g'_i : 1 \leq i \leq n - 1\}$, Where $e_i = u_i u'_i, e'_i = u'_i u_{i+1}, f_i = u_i v''_i, f'_i = v'_i u_{i+1}, g_i = v_i v'_i$, and $g'_i = v''_i v_{i+1}$.

From the lemma 2.5, $\chi'_e(S(S'(P_n))) \geq \Delta(S(S'(P_n))) \geq 4$.

To prove the upper bound, $\chi'_e(S(S'(P_n))) \leq \Delta(S(S'(P_n))) \leq 4$.

Consider a mapping $c : E(S(S'(P_n))) \rightarrow \{1, 2, 3, 4\}$ as follows. The edge set $E(S(S'(P_n)))$ are partition into four equitable edge coloring has given below. We consider the following two cases

Case (i): When n is odd

$$E_1 = \{e_1, e_3, \dots, e_{n-2}\} \cup \{f_2, f_4, \dots, f_{n-1}\} \cup \{g_1, g_3, \dots, g_{n-2}\}$$

$$E_2 = \{e'_1, e'_3, \dots, e'_{n-1}\} \cup \{f'_2, f'_4, \dots, f'_{n-2}\} \cup \{g'_1, g'_3, \dots, g'_{n-1}\}$$

$$E_3 = \{e_2, e_4, \dots, e_{n-2}\} \cup \{f_1, f_3, \dots, f_{n-1}\} \cup \{g_2, g_4, \dots, g_{n-2}\}$$

$$E_4 = \{e'_2, e'_4, \dots, e'_{n-1}\} \cup \{f'_1, f'_3, \dots, f'_{n-2}\} \cup \{g'_2, g'_4, \dots, g'_{n-1}\}$$

Case (ii): When n is even

$$E_1 = \{e_1, e_3, \dots, e_{n-2}\} \cup \{f_2, f_4, \dots, f_{n-2}\} \cup \{g_1, g_3, \dots, g_{n-1}\}$$

$$E_2 = \{e'_1, e'_3, \dots, e'_{n-1}\} \cup \{f'_2, f'_4, \dots, f'_{n-2}\} \cup \{g'_1, g'_3, \dots, g'_{n-1}\}$$

$$E_3 = \{e_2, e_4, \dots, e_{n-2}\} \cup \{f_1, f_3, \dots, f_{n-1}\} \cup \{g_2, g_4, \dots, g_{n-2}\}$$

$$E_4 = \{e'_2, e'_4, \dots, e'_{n-2}\} \cup \{f'_1, f'_3, \dots, f'_{n-1}\} \cup \{g'_2, g'_4, \dots, g'_{n-2}\}$$

The edges of the graph $S(S'(P_n))$ are categorized into distinct independent color classes E_1, E_2, E_3 and E_4 . Also for example (See Figure 1), if $n = 6$, for which $E_1 = E_2 = E_3 = E_4 = 7$. The condition of equitable edge coloring $\| |E_i| - |E_j| \| \leq 1$ for $i \neq j$ is satisfied. Thus the graph $S(S'(P_n))$ is 4-equitably edge colored. This implies that $\chi'_e(S(S'(P_n))) \leq \Delta(S(S'(P_n))) \leq 4$. Hence $\chi'_e(S(S'(P_n))) = 4$.

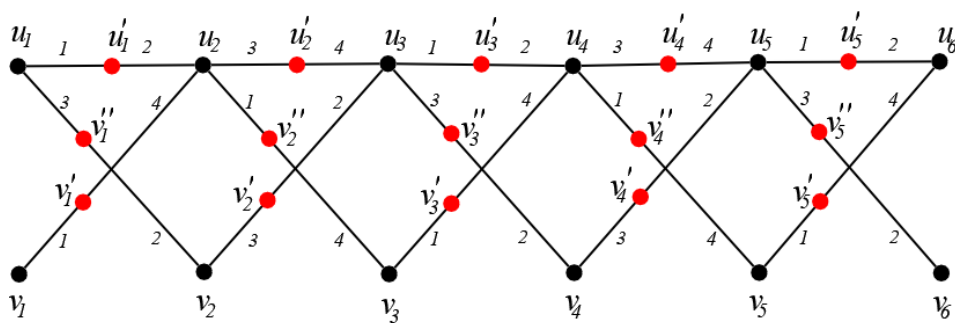


Figure 1: $S(S'(P_6))$ and its equitable edge coloring

Theorem 3.2: Let $S(S'(C_n))$ be the subdivision of splitting graph of cycle then $\chi'_e(S(S'(C_n))) = 4, n \geq 3$.

Proof: Let $S'(C_n)$ contains $2n$ vertices and $3n$ edges and its subdivision graph $S(S'(C_n))$ has following vertices and edges as follows,

Let $V(S(S'(C_n))) = \{u_i, u'_i, v_i : 1 \leq i \leq n\} \cup \{v'_i, v''_i : 1 \leq i \leq n-1\}$ and

$$E(S(S'(C_n))) = \{e_i, e'_i, f_i, f'_i, g_i, g'_i : 1 \leq i \leq n-1\}, \text{ Where } e_i = u_i u'_i, e'_i = u'_i u_{(i+1) \bmod n}, f_i = u_i v'_i, f'_i = v'_i u_{(i+1) \bmod n}, g_i = v_{i+1} v''_i, \text{ and } g'_i = v'_i v_i.$$

From the lemma 2.5, $\chi'_e(S(S'(C_n))) \geq \Delta(S(S'(C_n))) \geq 4$.

To prove the upper bound, $\chi'_e(S(S'(C_n))) \leq \Delta(S(S'(C_n))) \leq 4$.

Consider a mapping $c : E(S(S'(C_n))) \rightarrow \{1, 2, 3, 4\}$ as follows. The edge set $E(S(S'(C_n)))$ are partition into four equitable edge coloring has given below. We consider the following two cases

Case (i): When n is even

$$E_1 = \{e_1, e_2, \dots, e_n\} \cup \{g_2, g_4, \dots, g_{n-2}\} \cup \{g_n\}$$

$$E_2 = \{e'_1, e'_2, \dots, e'_n\} \cup \{g'_1, g'_3, \dots, g'_{n-1}\}$$

$$E_3 = \{f_1, f_2, \dots, f_n\} \cup \{g'_2, g'_4, \dots, g'_n\}$$

$$E_4 = \{f'_1, f'_3, \dots, f'_n\} \cup \{g_1, g_3, \dots, g_{n-1}\}$$

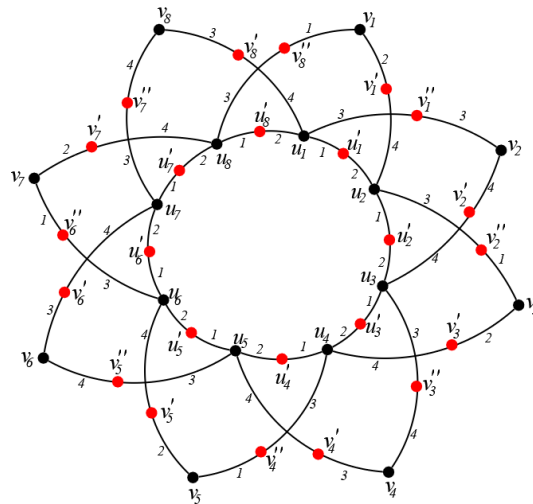


Figure 2: $S(S'(C_6))$ and its equitable edge coloring

Case (ii): When n is odd and for $i \in \{1, 2, 3, \dots, n\}$

$$E_1 = \{e_1, e_3, \dots, e_n\} \cup \{f'_1, f'_3, \dots, f'_{n-2}\} \cup \{g_2, g_4, \dots, g_{n-1}\} \cup \{g_n\}$$

$$E_2 = \{e'_1, e'_3, e'_5, \dots, e'_n\} \cup \{f_2, f_4, f_6, \dots, f_{n-1}\} \cup \{g'_1, g'_3, g'_5, \dots, g'_n\}$$

$$E_3 = \{e_2, e_4, e_6, \dots, e_{n-1}\} \cup \{f_1, f_3, f_5, \dots, f_n\} \cup \{g'_2, g'_4, \dots, g'_{n-1}\}$$

$$E_4 = \{e'_2, e'_4, \dots, e'_{n-1}\} \cup \{f_2, f_4, \dots, f_{n-1}\} \cup \{f'_n\} \cup \{g_1, g_3, \dots, g_{n-2}\}$$

The edges of the graph $S(S'(C_n))$ are categorized into distinct independent color classes E_1, E_2, E_3 and E_4 . For example (See Figure 2, 3), if $n=5$, for which $|E_1|=|E_2|=|E_3|=|E_4|=7$ and if $n=5$, for which $|E_1|=|E_2|=|E_3|=|E_4|=12$. The condition of equitable edge coloring $||E_i|-|E_j|| \leq 1$ for $i \neq j$ is satisfied. Thus, the graph $S(S'(C_n))$ is 4-equitably edge colored. This implies that $\chi'_e(S(S'(C_n))) \leq \Delta(S(S'(C_n))) \leq 4$. Hence $\chi'_e(S(S'(C_n))) = 4$.

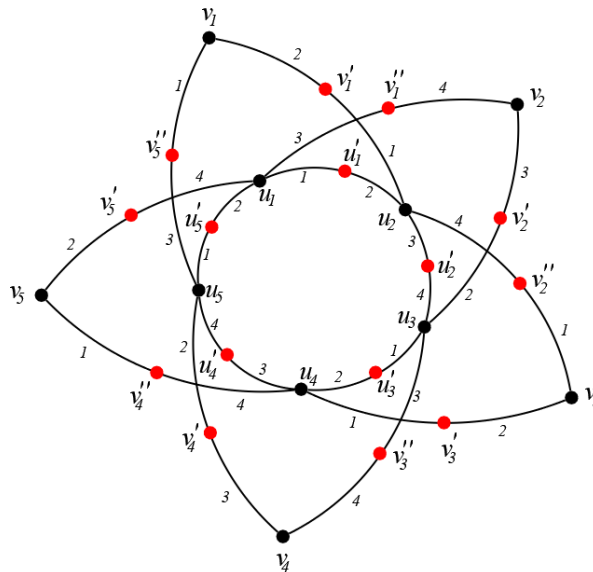


Figure 3: $S(S'(C_5))$ and its equitable edge coloring

Theorem 3.3: Let $S(S'(K_{1,n}))$ be the subdivision of splitting graph of star, then $\chi'_e(S(S'(K_{1,n}))) = 2n, n \geq 3$.

Proof: Let $S'(K_{1,n})$ contains $2n+2$ vertices and $3n$ edges and its subdivision graph $S(S'(K_{1,n}))$ has contains of $5n+2$ vertices and $6n$ edges as follows:

Let $V(S(S'(K_{1,n}))) = \{u, u', u_i, u'_i, v_i, v'_i, v''_i : 1 \leq i \leq n\}$ and

$E(S(S'(K_{1,n}))) = \{e_i, e'_i, f_i, f'_i, g_i, g'_i : 1 \leq i \leq n\}$, Where $e_i = u'v''_i, e'_i = u''_iv'_i, f_i = u''_iv'_i, f'_i = uv''_i, g_i = uv''_i$, and $g'_i = v''_iu'_i$.

From the Lemma 2.5, $\chi'_e(S(S'(K_{1,n}))) \geq \Delta(S(S'(K_{1,n}))) \geq 4$.

To prove the upper bound, $\chi'_e(S(S'(K_{1,n}))) \leq \Delta(S(S'(K_{1,n}))) \leq 4$.

Consider a mapping $c : E(S(S'(K_{1,n}))) \rightarrow \{1, 2, 3, 4\}$ as follows. The edge set $E(S(S'(K_{1,n})))$ are partition into four equitable edge coloring has given below.

$$\begin{aligned}
 E_1 &= \{e_1, f_1, g_1\} \\
 E_2 &= \{e_2, f_2, g_2\} \\
 E_3 &= \{e_3, f_3, g_3\} \\
 E_4 &= \{e_4, f_4, g_4\} \\
 E_5 &= \{e_5, f_5, g_5\} \\
 &\dots\dots \\
 &\dots\dots \\
 E_{n-2} &= \{e_{n-2}, f_{n-2}, g_{n-2}\} \\
 E_{n-1} &= \{e_{n-1}, f_{n-1}, g_{n-1}\} \\
 E_n &= \{e_n, f_n, g_n\} \\
 E_{n+1} &= \{e_{n+1}, f_{n+1}, g_{n+1}\} \\
 E_{n+2} &= \{e_{n+2}, f_{n+2}, g_{n+2}\} \\
 &\dots\dots \\
 &\dots\dots \\
 E_{2n-1} &= \{e'_{n-1}, f'_{n-1}, g'_{n-1}\} \\
 E_{2n} &= \{e'_n, f'_n, g'_n\}
 \end{aligned}$$

The edges of the graph $S(S'(K_{1,n}))$ are categorized into distinct independent color classes $E_1, E_2, E_3, E_4, \dots, E_{2n}$. For example (See Figure 4), if $n = 6$, for which $|E_1| = |E_2| = \dots = |E_{2n-1}| = |E_{2n}| = 3$. The condition of equitable edge coloring $||E_i| - |E_j|| \leq 1$ for $i \neq j$ is satisfied. Thus, the graph $S(S'(K_{1,n}))$ is equitable edge colored with $2n$ colors. This implies that $\chi'_e(S(S'(K_{1,n}))) \leq 2n$. Hence $\chi'_e(S(S'(K_{1,n}))) = 2n$.

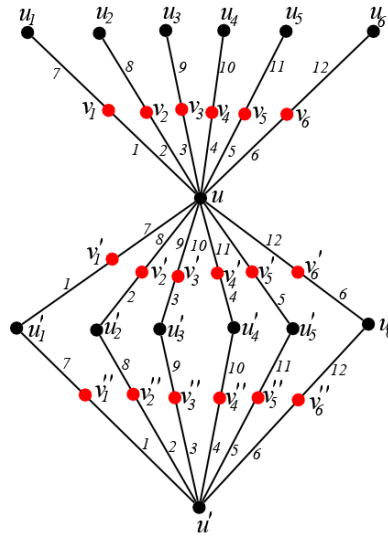


Figure 4: $S(S'(K_{1,6}))$ and its equitable edge coloring

Theorem 3.4: Let $S(D_2(P_n))$ be the subdivision of splitting graph for the shadow graph of path, then $\chi'_e(S(D_2(P_n))) = 4, n \geq 3$.

Proof: Let $D_2(P_n)$ contains $2n$ vertices and $3n+2$ edges and its subdivision graph T has vertex set and edge set as follows:

Let $V(S(D_2(P_n))) = \{u_i, v_i : 1 \leq i \leq n\} \cup \{u'_i, u''_i, v'_i, v''_i : 1 \leq i \leq n-1\}$ and

$$E(S(D_2(P_n))) = \{e_i, e'_i, f_i, f'_i, g_i, g'_i : 1 \leq i \leq n-1\}, \text{ Where } e_i = u_i u'_i, e'_i = u'_i u_{i+1}, f_i = u_i v''_i, f'_i = v'_i u_{i+1}, g_i = v_i v'_i, g'_i = v''_i v_{i+1}, h_i = v_i u''_i \text{ and } h'_i = u''_i v_{i+1}.$$

From the Lemma 2.5, $\chi'_e(S(D_2(P_n))) \geq \Delta(S(D_2(P_n))) \geq 4$.

To prove the upper bound, $\chi'_e(S(D_2(P_n))) \leq \Delta(S(D_2(P_n))) \leq 4$,

Consider a mapping $c : E(S(D_2(P_n))) \rightarrow \{1, 2, 3, 4\}$ as follows. The edge set $E(S(D_2(P_n)))$ are partition into four equitable edge coloring has given below.

For $i \in \{1, 2, 3, \dots, n\}$

$$\begin{aligned} E_1 &= \{e_1, e_3, \dots, e_{n-1}\} \cup \{f_2, f_4, \dots, f_{n-2}\} \cup \{g_1, g_3, \dots, g_{n-1}\} \cup \{h_2, h_4, \dots, h_{n-2}\} \\ E_2 &= \{e'_1, e'_3, \dots, e'_{n-1}\} \cup \{f'_2, f'_4, \dots, f'_{n-2}\} \cup \{g'_1, g'_3, \dots, g'_{n-1}\} \cup \{h'_2, h'_4, \dots, h'_{n-2}\} \\ E_3 &= \{e_2, e_4, \dots, e_{n-2}\} \cup \{f_1, f_3, \dots, f_{n-1}\} \cup \{g_2, g_4, \dots, g_{n-2}\} \cup \{h_1, h_3, \dots, h_{n-1}\} \\ E_4 &= \{e'_2, e'_4, \dots, e'_{n-2}\} \cup \{f'_1, f'_3, \dots, f'_{n-1}\} \cup \{g'_2, g'_4, \dots, g'_{n-2}\} \cup \{h'_1, h'_3, \dots, h'_{n-1}\} \end{aligned}$$

The edges of the graph $S(D_2(P_n))$ are categorized into distinct independent color classes E_1, E_2, E_3 and E_4 . Also for example (See Figure 1), if $n = 6$, for which $E_1 = E_2 = E_3 = E_4 = 10$. The condition of equitable edge coloring $\| |E_i| - |E_j| \| \leq 1$ for $i \neq j$ is satisfied. Thus the graph $S(D_2(P_n))$ is equitable edge colored with 4 colors. This implies that $\chi'_e(S(D_2(P_n))) \leq \Delta(S(D_2(P_n))) \leq 4$. Hence $\chi'_e(S(D_2(P_n))) = 4$.

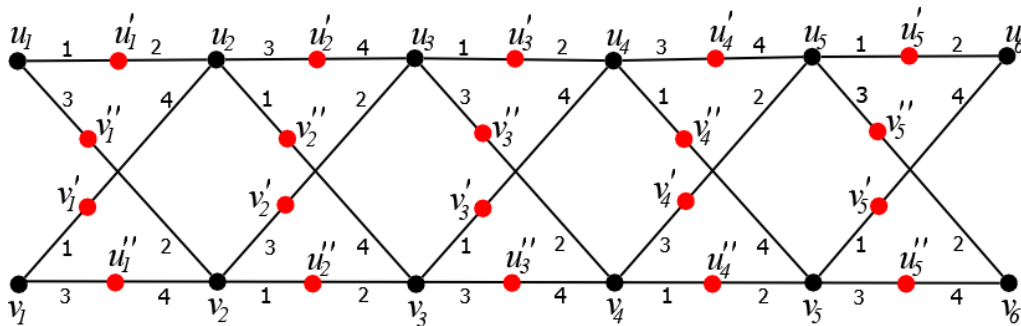


Figure 5: $S(D_2(P_6))$ and its equitable edge coloring

Theorem 3.5: Let $S(D_2(C_n))$ be the subdivision of splitting graph for the shadow graph of path, then $\chi'_e(S(D_2(C_n))) = 4, n \geq 3$.

Proof: Let $D_2(C_n)$ contains $2n$ vertices and $4n$ edges and its subdivision graph $S(D_2(C_n))$ has vertex set and edge set as follows:

Let $V(S(D_2(C_n))) = \{u_i, v_i, u'_i, u''_i, v'_i, v''_i : 1 \leq i \leq n\}$ and

$E(S(D_2(C_n))) = \{e_i, e'_i, f_i, f'_i, g_i, g'_i : 1 \leq i \leq n\}$, , Where $e_i = u_i u'_i, e'_i = u'_i u_{(i+1) \bmod n}, f_i = u_i v'_i, f'_i = v'_i u_{(i+1) \bmod n}, g_i = v_i v'_i, g'_i = v'_i v_{(i+1) \bmod n}, h_i = v_i u''_i$ and $h'_i = u''_i v_{(i+1) \bmod n}$.

From the Lemma 2.5, $\chi'_e(S(D_2(C_n))) \geq \Delta(S(D_2(C_n))) \geq 4$.

To prove the upper bound, $\chi'_e(S(D_2(C_n))) \leq \Delta(S(D_2(C_n))) \leq 4$.

Consider a mapping $c : E(S(D_2(C_n))) \rightarrow \{1, 2, 3, 4\}$ as follows. The edge set $E(S(D_2(C_n)))$ are partition into four equitable edge coloring has given below.

Case (i): When n is even

$$E_1 = \{e_1, e_2, \dots, e_n\} \cup \{h_1, h_2, \dots, h_n\}$$

$$E_2 = \{e'_1, e'_2, \dots, e'_n\} \cup \{h'_1, h'_2, \dots, h'_n\}$$

$$E_3 = \{f_1, f_3, \dots, f_{n-1}\} \cup \{g_1, g_2, \dots, g_n\}$$

$$E_4 = \{f'_1, f'_3, \dots, f'_n\} \cup \{g'_2, g'_4, \dots, g'_n\}$$

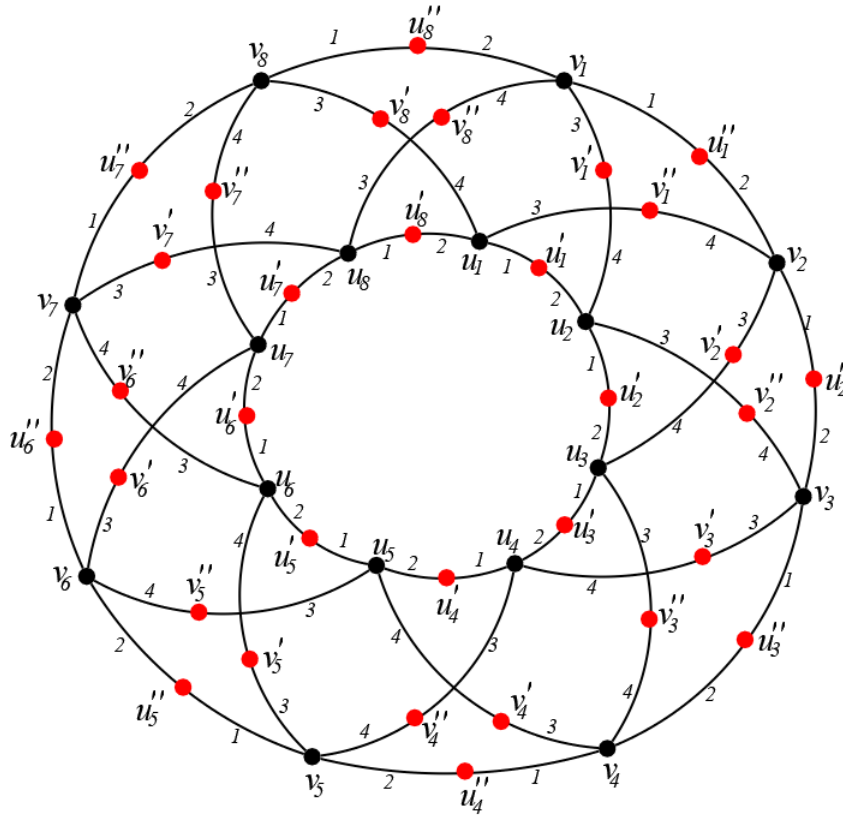


Figure 6: $S(D_2(C_6))$ and its equitable edge coloring

Case (ii): When n is odd

$$E_1 = \{e_1, e_3, \dots, e_n\} \cup \{f_2, f_4, \dots, f_{n-2}\} \cup \{g_1, g_3, \dots, g_{n-2}\} \cup \{h_2, h_4, \dots, h_n\}$$

$$E_2 = \{e'_1, e'_3, \dots, e'_n\} \cup \{f_3, f_5, \dots, f_n\} \cup \{g_3, g_5, \dots, g_n\} \cup \{h'_1, h'_3, \dots, h'_n\}$$

$$E_3 = \{f_1, g_1\} \cup \{e_2, e_4, \dots, e_{n-1}\} \cup \{f'_2, f'_4, \dots, f'_{n-1}\} \cup \{g'_2, g'_4, \dots, g'_{n-1}\} \cup \{h_2, h_4, \dots, h_{n-1}\}$$

$$E_4 = \{e'_2, e'_4, \dots, e'_{n-1}\} \cup \{f_2, f_4, \dots, f_{n-1}\} \cup \{f'_n, g'_n\} \cup \{g_2, g_4, \dots, g_{n-1}\} \cup \{h'_2, h'_4, \dots, h'_{n-1}\}$$

The edges of the graph $S(D_2(C_n))$ are categorized into distinct independent color classes E_1, E_2, E_3 and E_4 . For example (See Figure 6, 7), if $n = 8$, for which $E_1 = E_2 = E_3 = E_4 = 16$; and if $n = 5$, then $E_1 = E_2 = E_3 = E_4 = 10$. The condition of equitable edge coloring $||E_i| - |E_j|| \leq 1$ for $i \neq j$ is satisfied. Thus the graph $S(D_2(C_n))$ is equitable edge colored with 4 colors. This implies that $\chi'_e(S(S(D_2(C_n)))) \leq \Delta(S(D_2(C_n))) \leq 4$. Hence $\chi'_e(S(D_2(C_n))) = 4$.

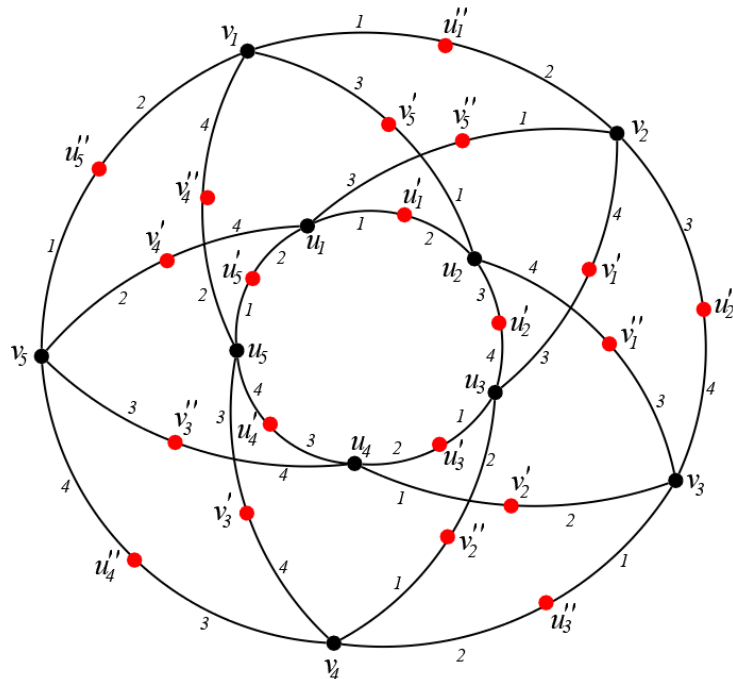


Figure 7: $S(D_2(C_5))$ and its equitable edge coloring

Theorem 3.6: Let $S(D_2(K_{1,n}))$ be the subdivision of shadow graph of star, then $\chi'_e(S(D_2(K_{1,n}))) = 2n, n \geq 3$.

Proof: Let $D_2(K_{1,n})$ contains $2n+2$ vertices and $4n$ edges and its subdivision graph $S(D_2(K_{1,n}))$ has contains of $5n+2$ vertices and $6n$ edges as follows:

Let $V(S(D_2(K_{1,n}))) = \{u, v, u_i, u'_i, u''_i, v_i, v'_i, v''_i : 1 \leq i \leq n\}$ and

$E(S(D_2(K_{1,n}))) = \{e_i, e'_i, e''_i, e'''_i, f_i, f'_i, f''_i, f'''_i : 1 \leq i \leq n\}$, Where $e_i = uv''_i, e'_i = uu'_i, e''_i = u_iu'_i, e'''_i = u_iu''_i, f_i = vu''_i, f'_i = vv'_i, f''_i = v_iv'_i, \text{ and } f'''_i = v_iv''_i$.

From the Lemma 2.5, $\chi'_e(S(D_2(K_{1,n}))) \geq \Delta(S(D_2(K_{1,n}))) \geq 4$.

To prove the upper bound, $\chi'_e(S(D_2(K_{1,n}))) \leq \Delta(S(D_2(K_{1,n}))) \leq 4$.

Consider a mapping $c : E(S(D_2(K_{1,n}))) \rightarrow \{1, 2, \dots, n\}$ as follows. The edge set $E(S(D_2(K_{1,n})))$ are partition into four equitable edge coloring has given below.

$$E_1 = \{e_1, e''_1, f_1, f'''_1\}$$

$$E_2 = \{e_2, e'''_1, f'_1, f'''_1\}$$

$$E_3 = \{e_3, e''_2, f_2, f'''_2\}$$

$$\begin{aligned}
 E_4 &= \{e_4, e_2''', f_2', f_2'''\} \\
 E_5 &= \{e_3, e_3'', f_3, f_3''\} \\
 &\dots\dots \\
 &\dots\dots \\
 E_{n-1} &= \{e_{n-1}, e_{n-3}''', f_{n-3}', f_{n-3}'''\} \\
 E_n &= \{e_n, e_{n-2}'', f_{n-2}', f_{n-2}''\} \\
 E_{n+1} &= \{e_1', e_3'', f_3', f_3''\} \\
 E_{n+2} &= \{e_2', e_4'', f_4', f_4''\} \\
 E_{n+3} &= \{e_3', e_4''', f_4', f_4'''\} \\
 E_{n+4} &= \{e_4', e_5'', f_5', f_5''\} \\
 E_{n+5} &= \{e_5', e_5''', f_5', f_5'''\} \\
 &\dots\dots \\
 &\dots\dots \\
 E_{2n-4} &= \{e_{n-4}', e_{n-2}''', f_{n-2}', f_{n-2}'''\} \\
 E_{2n-3} &= \{e_{n-3}', e_{n-1}'', f_{n-1}', f_{n-1}''\} \\
 E_{2n-2} &= \{e_{n-2}', e_{n-1}''', f_{n-1}', f_{n-1}'''\} \\
 E_{2n-1} &= \{e_{n-1}', e_n'', f_n', f_n''\} \\
 E_{2n} &= \{e_n', e_n''', f_n', f_n'''\}
 \end{aligned}$$

The edges of the graph $S(D_2(K_{1,n}))$ are categorized into distinct independent color classes $E_1, E_2, E_3, E_4, \dots, E_{2n}$. For example (See Figure 8), if $n = 6$, for which $|E_1| = |E_2| = \dots = |E_{2n-1}| = |E_{2n}| = 4$. The condition of equitable edge coloring $||E_i| - |E_j|| \leq 1$ for $i \neq j$ is satisfied. Thus, the graph $S(D_2(K_{1,n}))$ is equitable edge colored with $2n$ colors. This implies that $\chi_e'(S(D_2(K_{1,n}))) \leq 2n$. Hence $\chi_e'(S(D_2(K_{1,n}))) = 2n$.

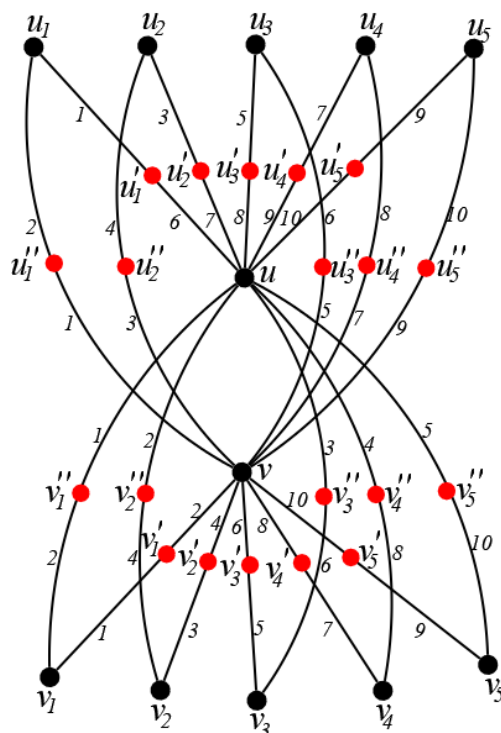


Figure 8: $S(D_2(K_{1,6}))$ and its equitable edge coloring

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