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# On Total Coloring of Triple Star and Lobster Graphs

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**Abstract**

A *k-total coloring* of a graph *G* is an assignment of *k* colors to the elements (vertices and edges) of *G* such that adjacent or incident elements have different colors. The total chromatic number is the smallest integer *k* for which *G* has a k-total coloring. The well-known Total Coloring Conjecture asserts that the total chromatic number of a graph is either Δ(*G*) + 1 or Δ(*G*) + 2, where Δ(*G*) is the maximum degree of *G*. In this paper, we consider the triple star graph, lobster graph and its line, middle, total graphs and also splitting graph of triple star. We obtained the preceding graphs has total chromatic number equal to Δ(G) + 1.

**Keywords**: Total coloring, total chromatic number, triple star graph, lobster graph, line, middle, total, splitting graph.

**AMS subject classification:** 05C15.

## 1. Introduction

The graph described is finite, undirected, and has no loops or multiple edges. A $k$ *-total coloring* is an assignment of $k$ colors to the elements (ie, vertices and edges) of $G$, for which two adjacent or incident elements have distinct colors. The *total chromatic number* $\chi''(G)$, is the least $k$, for which $G$ has a $k$ -total coloring. Behzad [2] and vizing [17] independently, suggested the famous conjecture, known as TCC: for any graph $G$, $\Delta(G)+1 \le \chi''(G) \le \Delta(G)+2.$ This conjecture was authenticated by Rosenfeld [15] and Vijayadithya [16] for $\Delta = 3,$ and by kostochka [8, 9] for $\Delta \le 5.$ This result was firstly given in [3] for $\Delta \ge 14.$ In [10], this result was protracted to $\Delta \ge 9.$ A survey paper on total coloring would likely include a comprehensive collection of articles covering various aspects of total coloring [4]. Jayaraman et al. [7, 11, 12, 13, 14] discussed the total coloring of line, middle, total and splitting graph of double star graph, snake graph families, splitting graph of path, cycle and star graph and certain convex polytope graphs

Total coloring provides a powerful tool for modeling and solving various practical problems arising in scheduling, wireless networks, telecommunications. In [1], the author has extended the concept of the double star graph to introduce the triple star graph. In this paper, we obtained the total chromatic number of triple star, lobster graph and its line, middle, total graph and also splitting graph of triple star graph.

## 2. Preliminaries

**Definition 2.1** *Triple star graph* $\zeta_\eta = K_{1,\eta,\eta,\eta}$ [1] is constructed from the double star graph $K_{1,\eta,\eta}$ by adding a new pendent edge to each existing pendant node, then it forms a tree structure. It has $3\eta+1$ nodes and $3\eta$ edges.

**Definition 2.2.** The *lobster graph* $\gamma_\eta = L_\eta(2,r)$ [6] is constructed by the path on $\eta$ nodes as a backbone. Each node in the backbone is attached to two distinct node hands, and each node hand is connected to $r$ distinct node fingers, each of which has degree one, throughout this article, we take $r=1.$

**Definition 2.3.** The *line graph* $L(G)$ [7] has nodes that are edges of $G,$ and two nodes in

$L(G)$ are adjacent, whenever their edges are adjacent in $G.$

**Definition 2.4.** The *middle graph* $M(G)$ [5, 13] is formed by subdividing each edge exactly once and connecting these newly obtained nodes of adjacent edges of $G.$

**Definition 2.5.** The *total graph* $T(G)$ [13] is a middle graph, adding an edge between the vertices whenever they are adjacent in $G$ .

**Definition 2.6.** The *splitting graph* $S(G)$ [12] is obtained by adding a new vertex $v^{'}$ corresponding to each vertex $v$ of $G$ such that $N(v)=N(v^{'}).$

**Lemma 2.7.** [18] For any simple graph $G,$ $\chi^{''}(G) \geq \Delta(G)+1.$

**Theorem 2.8.** [18] Let $K_n$ be the complete graph, then $\chi^{''}(K_n) = \begin{cases} n, & if\ n\ is\ odd \\ n+1, & if\ n\ is\ even \end{cases}$

## 3. Results and Discussion

**Theorem 3.1.** For any $\eta \geq 4$ , $\chi^{''}(\zeta_\eta) = \eta+1$ .

Proof: Let $V(\zeta_\eta) = \{u\} \bigcup \{u_\xi : 1 \leq \xi \leq \eta\} \bigcup \{v_\xi ; 1 \leq \xi \leq \eta\} \bigcup \{w_\xi ; 1 \leq \xi \leq \eta\}$ , where $u$ is the root vertex of the triple star graph. $E(\zeta_\eta) = \{e_\xi = uu_\xi, f_\xi = u_\xi v_\xi, e_\xi = v_\xi w_\xi : 1 \leq \xi \leq \eta\}$

Define total coloring $\varphi$, such that $\varphi : V(\zeta_\eta) \bigcup E(\zeta_\eta) \rightarrow \{1,2,3,...,\eta+1\}$

For $1 \leq \xi \leq \eta$

$\varphi(u_\xi) = \eta+1$ ; $\varphi(u_\xi) = \varphi(w_\xi) = \xi+1(\text{mod } \eta)$ ;

$\varphi(v_\xi) = \xi+4(\text{mod } \eta)$ ; $\varphi(e_\xi) = \varphi(g_\xi) = \xi(\text{mod } \eta)$ ;

$\varphi(f_\xi) = \xi+3(\text{mod } \eta)$

By this procedure, the graph $\zeta_\eta$ is attained with $\eta+1$ total colorable. So, it is clear that $\chi^{''}(\zeta_\eta) \leq \eta+1.$ Since $\Delta(\zeta_\eta) = \eta$ and by lemma 2.7, it follows that $\chi^{''}(\zeta_\eta) \geq \Delta(\zeta_\eta)+1 \geq \eta+1.$ Therefore $\chi^{''}(\zeta_\eta) = \eta+1.$ Equivalently, this is true for all other values of $\eta \geq 4.$ Hence $\chi^{''}(\zeta_\eta) = \eta+1.$

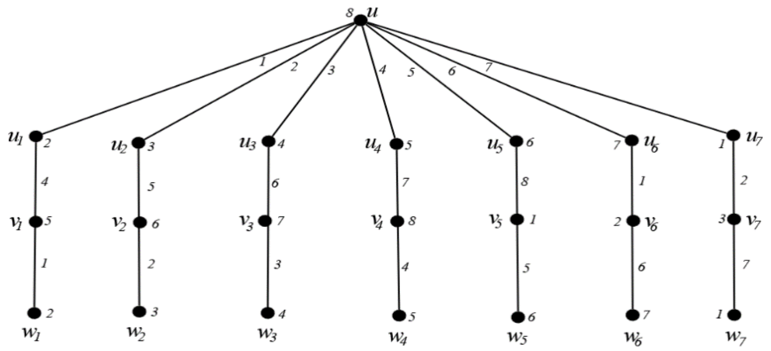


Fig 1: Total coloring of triple star graph $\zeta_7$

**Theorem 3.2:** For any $\eta \geq 4$, $\chi''(L(\zeta_\eta)) = \eta + 1$.

**Proof:** Let $V(L(\zeta_\eta)) = \{u_\xi, v_\xi, w_\xi : 1 \leq \xi \leq \eta\}$ and

$E(L(\zeta_\eta)) = \{u_\xi v_\xi, v_\xi w_\xi : 1 \leq \xi \leq \eta\} \cup \{u_\xi v_\gamma : 1 \leq \xi \leq \eta, \gamma > \xi, \xi + 1 \leq \gamma \leq \eta\}$

Define $\varphi$, such that $\varphi : V(L(\zeta_\eta)) \cup E(L(\zeta_\eta)) \to \{1, 2, 3, ..., \eta + 1\}$.

$\varphi(u_\xi) = \xi,$ for all $1 \leq \xi \leq \eta$; $\quad \varphi(v_\xi) = \begin{cases} \eta + 1, & if \ \ \xi = 1, \eta \\ \xi - 1, & if \ \ 2 \leq \xi \leq \eta - 1 \end{cases}$;

$\varphi(w_\xi) = \begin{cases} 2 + \xi, & if \ \ 1 \leq \xi \leq \eta - 1 \\ 1, & if \ \ \xi = \eta \end{cases}$ $\quad \varphi(u_\xi v_\xi) = \begin{cases} 2\xi, & if \ \ 2\xi \not\equiv 0 \ (\mathrm{mod}\, \eta + 1) \\ \eta + 1, & \text{otherwise} \end{cases}$;

$\varphi(v_\xi w_\xi) = \xi,$ for all $1 \leq \xi \leq \eta$; $\quad \varphi(u_\xi u_\gamma) = \begin{cases} \xi + \gamma, & if \ \ (\xi + \gamma) \not\equiv 0 \ (\mathrm{mod}\, \eta + 1) \\ \eta + 1, & \text{otherwise } \gamma > \xi \ , \ \xi + 1 \leq \gamma \leq \eta \end{cases}$

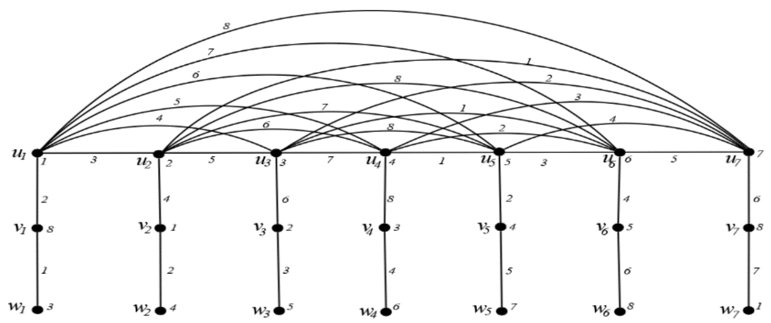


Fig 2: Total coloring of $L(\zeta_7)$

By this procedure, the graph $L(\zeta_\eta)$ is attained with $\eta + 1$ total colorable. So, it is clear that $\chi''(L(\zeta_\eta)) \leq \eta + 1$. Since $\Delta(L(\zeta_\eta)) = \eta$ and by lemma2.7, it follows that $\chi''(L(\zeta_\eta)) \geq \Delta(L(\zeta_\eta)) + 1$ $\geq \eta + 1$. Therefore $\chi''(L(\zeta_\eta)) = \eta + 1$. Equivalently, this is true for all other values of $\eta \geq 4$. Hence $\chi''(L(\zeta_\eta)) = \eta + 1$.

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**Theorem 3.3:** For any $\eta \geq 4$, $\chi''(M(\zeta_\eta)) = \eta + 3$.

Proof: $V(M(\zeta_\eta)) = \left\{\{v\} \cup \{v'_\xi\} \cup \{v_\xi\} \cup \{u'_\xi\} \cup \{u_\xi\} \cup \{w'_\xi\} \cup \{w_\xi\} : 1 \leq \xi \leq \eta\right\}$ and

$$E(M(\zeta_\eta)) = \left\{vv'_\xi; v'_\xi v_\xi; v_\xi u'_\xi; u'_\xi u_\xi; u_\xi w'_\xi; w'_\xi w_\xi; w'_\xi u'_\xi; u'_\xi v'_\xi : 1 \leq \xi \leq \eta\right\} \cup \left\{v'_\xi v'_\gamma : 1 \leq \xi \leq t-1, \xi+1 \leq \gamma \leq \eta\right\}$$

The vertices $u, u_\xi (1 \leq \xi \leq \eta)$ induced a clique of order $\eta + 3$ in $M(\zeta_\eta)$. Define $\varphi$, such that $\varphi : V(M(\zeta_\eta)) \cup E(M(\zeta_\eta)) \to \{1, 2, 3..., \eta + 3\}$. Consider the following two cases

**Case (i):** When $\eta$ is even, for $1 \leq \xi \leq \eta$

$\varphi(v) = \eta + 3$; $\varphi(v_\xi) = \eta + 2, \varphi(u_\xi) = \eta$; $\varphi(v'_\xi) = \xi$; $\varphi(u'_\xi) = \eta + 1$; $\varphi(w'_\xi) = \eta + 3$;

$\varphi(w_\xi) = \eta + 2$; $\varphi(v'_\xi v_\xi) = \begin{cases} \eta, & \xi = 1 \\ \xi - 1, & 2 \leq \xi \leq \eta \end{cases}$; $\varphi(v_\xi u'_\xi) = \eta$; $\varphi(u'_\xi u_\xi) = \eta + 3$;

$\varphi(u_\xi w'_\xi) = \eta - 2$; $\varphi(w'_\xi w_\xi) = \eta - 3$; $\varphi(w'_\xi u'_\xi) = \eta - 1$; $\varphi(u'_\xi v'_\xi) = \eta + 2$

$\varphi(vv'_\xi) = 2\xi,\ if\ \ 2\xi \not\equiv 0 \pmod{\eta + 3}$ ; $\varphi(v'_\xi v'_\gamma) = \begin{cases} \xi + \gamma,\ if\ \ (\xi + \gamma) \not\equiv 0 \pmod{\eta + 3}\ for\ 1, 2, ..., \eta - 1 \\ \eta + 3,\ otherwise\ \gamma > \xi\ ,\ \xi + 1 \leq \gamma \leq \eta \end{cases}$

**Case (ii):** When $\eta$ is odd, for $1 \leq \xi \leq \eta$

$\varphi(v) = \eta + 3$; $\varphi(v_\xi) = \eta + 2, \varphi(u_\xi) = \eta$; $\varphi(v'_\xi) = \xi$; $\varphi(u'_\xi) = \eta + 1$; $\varphi(w'_\xi) = \eta + 3$;

$\varphi(w_\xi) = \eta + 2$; $\varphi(v'_\xi v_\xi) = \begin{cases} \eta + 3, & \xi = 1 \\ \xi - 1, & 2 \leq \xi \leq \eta \end{cases}$; $\varphi(v_\xi u'_\xi) = \eta$; $\varphi(u'_\xi u_\xi) = \eta + 2$;

$\varphi(u_\xi w'_\xi) = \eta - 2$; $\varphi(w'_\xi w_\xi) = \eta - 3$; $\varphi(w'_\xi u'_\xi) = \eta - 1$; $\varphi(u'_\xi v'_\xi) = \eta + 2$;

$\varphi(vv'_\xi) = 2\xi,\ if\ \ 2\xi \not\equiv 0 \pmod{\eta + 2}$ ; $\varphi(v'_\xi v'_\gamma) = \begin{cases} \xi + \gamma,\ if\ \ (\xi + \gamma) \not\equiv 0 \pmod{\eta + 2}\ for\ 1, 2, ..., \eta - 1 \\ \eta + 2,\ \text{otherwise}\ \gamma > \xi\ ,\ \xi + 1 \leq \gamma \leq \eta \end{cases}$

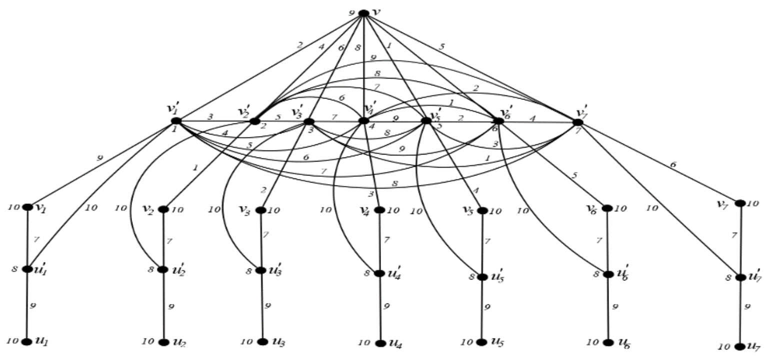

Fig 3: Total coloring of $M(\zeta_7)$

By this procedure, the graph $M(\zeta_\eta)$ is attained with $\eta+3$ total colorable. So, it is clear that $\chi''(M(\zeta_\eta)) \le \eta+3$. Since $\Delta(M(\zeta_\eta)) = \eta$ and by lemma 2.7, it follows that $\chi''(M(\zeta_\eta)) \ge \Delta(M(\zeta_\eta))+1$ $\ge \eta+3$. Therefore $\chi''(M(\zeta_\eta)) = \eta+3$. Equivalently, this is true for all other values of $\eta \ge 4$. Hence $\chi''(M(\zeta_\eta)) = \eta+3$.

**Theorem 3.4:** For any $\eta \ge 4$, $\chi''(T(\zeta_\eta)) = 2\eta+1$.

**Proof:** Let $V(T(\zeta_\eta)) = \left\{\{v\} \cup \{u'_\xi, v'_\xi, w'_\xi\} \cup \{u_\xi, v_\xi, w_\xi\} : 1 \le \xi \le \eta\right\}$ and

$$E(T(\zeta_\eta)) = \left\{vv'_\xi; v'_\xi v_\xi; v_\xi u'_\xi; u'_\xi u_\xi; u_\xi w'_\xi; w'_\xi w_\xi; w'_\xi u'_\xi; u'_\xi v'_\xi; u_\xi v_\xi; vv_\xi; w_\xi u_\xi : 1 \le \xi \le \eta\right\} \cup$$
$$\left\{v'_\xi v'_\xi : 1 \le \xi \le \eta-1, \xi+1 \le \gamma \le \eta\right\}$$

The vertices $u, u_\xi (1 \le \xi \le \eta)$ induced a clique of order $\eta+3$ in $T(\zeta_\eta)$. Define $\varphi$, such that $\varphi : V(T(\zeta_\eta)) \cup E(T(\zeta_\eta)) \to \{1, 2, 3..., 2\eta+1\}$.

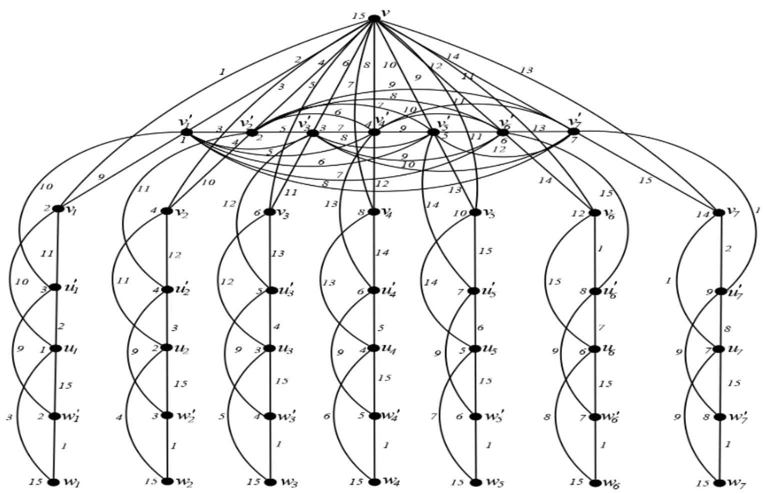

Fig 4: Total coloring of $T(\zeta_7)$

For $1 \le \xi \le \eta$

$\varphi(v) = 2\eta+1; \varphi(v'_\xi) = \xi; \varphi(v_\xi) = 2\xi;\ \varphi(u_\xi) = \xi; \varphi(w_\xi) = 2\eta+1;$

$$\varphi(u'_\xi) = \begin{cases} \xi+2, \ if \ \ \xi+2 \not\equiv 0 \ (\text{mod}\, \eta+2) \\ \eta+2, \ otherwise \end{cases}; \quad \varphi(w'_\xi) = \begin{cases} \xi+1, \ if \ \ \xi+1 \not\equiv 0 \ (\text{mod} \ \ \eta+1) \\ \eta+1, \ otherwise \end{cases};$$

$$\varphi(vv'_\xi) = \begin{cases} 2\xi, \ if \ \ 2\xi \not\equiv 0 \ (\text{mod}\, 2\eta) \\ 2\eta, \ otherwise \end{cases}; \ \varphi(v'_\xi v_\xi) = \begin{cases} \eta+1+\xi, \ if \ \ \eta+1+\xi \not\equiv 0 \ (\text{mod} \ \ 2\eta+1) \\ 2\eta+1, \ otherwise \end{cases}$$

$$\varphi(v'_\xi v'_\xi) = \begin{cases} \xi+\gamma, \ if \ \ (\xi+\gamma) \not\equiv 0 \ (\text{mod}\, 2\eta-1) \ for \ 1 \le \xi \le \eta-1 \\ 2\eta-1, \ otherwise \ \gamma > \xi \ , \ \xi+1 \le \gamma \le \eta \end{cases};$$

$\varphi(v_\xi u_\xi^{'}) = \begin{cases} \eta+3+\xi, \; if \;\; \eta+3+\xi \not\equiv 0 \text{ (mod } 2\eta+1) \\ 2\eta+1, \; otherwise \end{cases}$; $\varphi(u_\xi^{'} u_\xi) = \begin{cases} \xi+1, \; if \;\; \xi+1 \not\equiv 0 \text{ (mod } \eta+1) \\ \eta+1, \; otherwise \end{cases}$

$\varphi(u_\xi w_\xi^{'}) = 2\eta+1; \varphi(w_\xi^{'} w_\xi) = 1$; $\varphi(vv_\xi) = \begin{cases} 2\xi-1, \; if \;\; 2\xi-1 \not\equiv 0 \,\text{mod}(2\eta-1) \\ 2\eta-1, \; otherwise \end{cases}$;

$\varphi(u_\xi^{'} v_\xi^{'}) = \varphi(u_\xi v_\xi) = \begin{cases} \eta+2+\xi, \; if \;\; \eta+2+\xi \not\equiv 0 \;\; \text{mod}(2\eta+1) \\ 2\eta+1, \; otherwise \end{cases}$;

$\varphi(w_\xi^{'} u_\xi^{'}) = \eta+2$; $\varphi(w_\xi u_\xi) = 2+\xi$

By this procedure, the graph $T(\zeta_\eta)$ is attained with $2\eta+1$ total colorable. So, it is clear that $\chi^{''}(T(\zeta_\eta)) \le 2\eta+1$. Since $\Delta(T(\zeta_\eta)) = \eta$ and by lemma 2.7, it follows that $\chi^{''}(T(\zeta_\eta)) \ge \Delta(T(\zeta_\eta))+1$ $\ge \eta+3$. Therefore $\chi^{''}(T(\zeta_\eta)) = 2\eta+1$. Equivalently, this is true for all other values of $\eta \ge 4$. Hence $\chi^{''}(T(\zeta_\eta)) = 2\eta+1$.

**Theorem 3.5:** For any $\eta \ge 4$, $\chi^{''}(S(\zeta_\eta)) = 2\eta+1$.

**Proof:** Let $V(S(\zeta_\eta)) = \left\{ \{v\} \cup \{u_\xi^{'}, v_\xi^{'}, w_\xi^{'}\} \cup \{u_\xi, v_\xi, w_\xi\} : 1 \le \xi \le \eta \right\}$ and

$E(S(\zeta_\eta)) = \left\{ uu_\xi; u^{'}u_\xi; uu_\xi^{'}; u_\xi v_\xi; u_\xi^{'} v_\xi; u_\xi v_\xi^{'}; v_\xi w_\xi; v_\xi^{'} w_\xi; v_\xi w_\xi^{'} : 1 \le \xi \le \eta \right\}$

Define $\varphi$, such that $\varphi : V(S(\zeta_\eta)) \cup E(S(\zeta_\eta)) \to \{1, 2, 3 ... 2\eta+1\}$.

$\varphi(u) = \varphi(u^{'}) = 2\eta+1$; $\varphi(u_\xi) = \varphi(u_\xi^{'}) = \begin{cases} \xi+1, \; if \;\; (\xi+1) \not\equiv 0 \text{ (mod}\, \eta+1) \\ \eta+1, \; otherwise \end{cases}$;

$\varphi(v_\xi) = \varphi(v_\xi^{'}) = \eta+2$; $\varphi(w_\xi) = \varphi(w_\xi^{'}) = \eta+3$

$\varphi(uu_\xi) = \begin{cases} \eta+\xi, \; if \;\; \eta+\xi \not\equiv 0 \text{ (mod}\, 2\eta) \\ 2\eta, \; otherwise \end{cases}$; $\varphi(uu_\xi^{'}) = \varphi(u^{'}u_\xi) = \begin{cases} \xi, \; if \;\; \xi \not\equiv 0 \text{ (mod}\, \eta) \\ \eta, \; otherwise \end{cases}$;

$\varphi(u_\xi v_\xi) = \begin{cases} 2\eta, \; if \;\; 1 \le \xi \le \eta-1 \\ \eta-2, \;\; \xi = \eta \end{cases}$; $\varphi(v_\xi w_\xi) = \eta$; $\varphi(u_\xi v_\xi^{'}) = \varphi(u_\xi^{'} v_\xi) = 2\eta+1$;

$\varphi(v_\xi w_\xi^{'}) = 1$; $\varphi(v_\xi^{'} w_\xi) = 2$

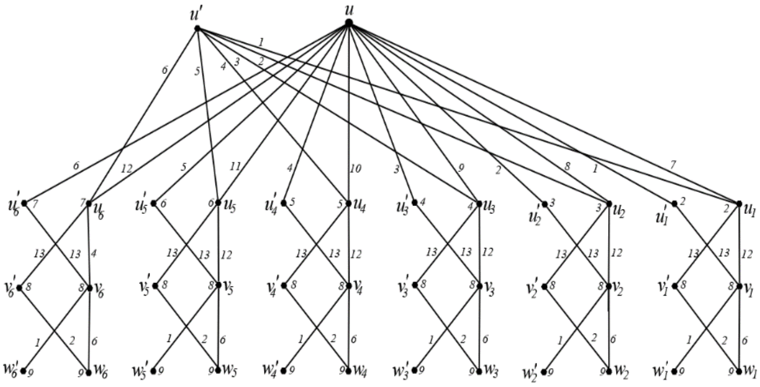


Fig 5: Total coloring of $S(\zeta_7)$

Based on the above procedure, the graph $S(\zeta_\eta)$ is attained with $2\eta+1$ total colorable. So, it is clear that $\chi''(S(\zeta_\eta)) \le 2\eta+1.$ Since $\Delta(S(\zeta_\eta)) = \eta$ and by lemma 2.7, it follows that $\chi''(S(\zeta_\eta)) \ge \Delta(S(\zeta_\eta))+1 \ge \eta+3.$ Therefore $\chi''(S(\zeta_\eta)) = 2\eta+1.$ Similarly, this result is true for all values of $\eta \ge 4.$ Hence $\chi''(S(\zeta_\eta)) = 2\eta+1.$

**Theorem 3.6**: let $\gamma_\eta = L_\eta(2,1)$ be the lobster, then $\chi''(\gamma_t) = 5.$

**Proof**: let $V(\gamma_\eta) = \{v_\xi, u_\xi, w_\xi, w'_\xi, u'_\xi : 1 \le \xi \le \eta\}$ and

$E(\gamma_\eta) = \{v_\xi u_\xi, u_\xi u'_\xi, v_\xi w_\xi, w_\xi w'_\xi : 1 \le \xi \le \eta\} \cup \{v_\xi v_{\xi+1} : 1 \le \xi \le \eta-1\}$

Define $\varphi : V(\gamma_\eta) \cup E(\gamma_\eta) \to \{1,2,3,4,5\}$ as follows.

The assigning of colors is given below:

For $1 \le \xi \le \eta$

$\varphi(u_\xi) = \varphi(v_\xi w_\xi) = 2,1;\ if\ \ \xi \equiv 1,0 \pmod 2$; $\varphi(v_\xi) = 1,2;\ if\ \ \xi \equiv 1,0 \pmod 2$;

$\varphi(w_\xi) = 3,4;\ if\ \ \xi \equiv 1,0 \pmod 2$; $\varphi(v_\xi u_\xi) = 5$; $\varphi(w_\xi w'_\xi) = 5$; $\varphi(u'_\xi) = 3$;

$\varphi(w'_\xi) = 2$; $\varphi(u_\xi u'_\xi) = 4$

For $1 \le \xi \le \eta-1$

$\varphi(v_\xi v_{\xi+1}) = 3,4;\ if\ \ \xi \equiv 1,0 \pmod 2$

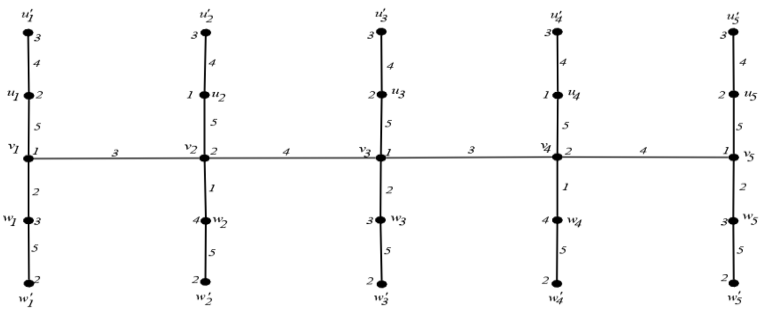

Fig 6: Total coloring of lobster graph $\gamma_5$

Based on the above coloring approach, the graph $\gamma_\eta$ is total colored with 5 colors. Thus $\chi''(\gamma_\eta) \le 5$. Since $\Delta(\gamma_\eta) = 4$ and by lemma 2.7, $\chi''(\gamma_\eta) \ge \Delta(\gamma_\eta) + 1 = 4 + 1 \ge 5$ and hence $\chi''(\gamma_\eta) = 5$.

**Theorem 3.7**: let $L(\gamma_\eta)$ be the line graph of lobster graph, then $\chi''(L(\gamma_\eta)) = 5$.

**Proof:** let $V(L(\gamma_\eta)) = \{u_\xi, w_\xi, w'_\xi, u'_\xi : 1 \le \xi \le \eta\} \bigcup \{v_\xi : 1 \le \xi \le \eta - 1\}$ and $E(L(\gamma_\eta)) = \{u_\xi w_\xi, u_\xi u'_\xi, w_\xi w'_\xi : 1 \le \xi \le \eta\} \bigcup \{v_\xi u_{\xi+1}, u_\xi v_\xi, w_\xi v_\xi, v_\xi w_{\xi+1} : 1 \le \xi \le \eta - 1\}$.

Define $\varphi : V(L(\gamma_\eta)) \bigcup E(L(\gamma_\eta)) \to \{1,2,3,4,5\}$ as follows. The assigning of colors is given below:

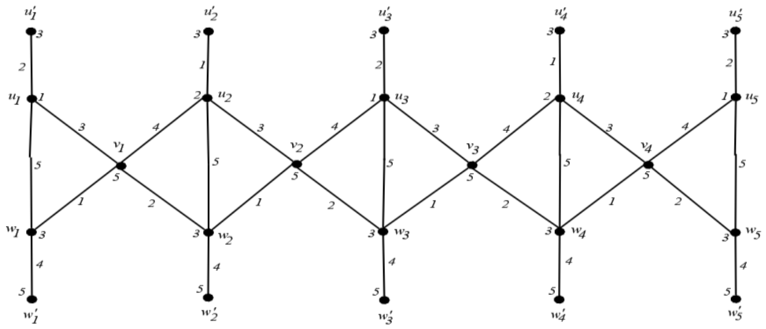

Fig 7: Total coloring of $L(\gamma_5)$

For $1 \le \xi \le \eta$

$\varphi(u_\xi) = 1,2;\ if\ \ \xi \equiv 1,0 \pmod 2$; $\varphi(u'_\xi u_\xi) = 2,1;\ if\ \ \xi \equiv 1,0 \pmod 2$; $\varphi(w_\xi) = 3$;

$\varphi(u'_\xi) = 3$; $\varphi(w'_\xi) = 5$; $\varphi(w_\xi w'_\xi) = 4$

For $1 \le \xi \le \eta - 1$

$\varphi(v_\xi) = 5$; $\varphi(u_\xi v_\xi) = 3$; $\varphi(v_\xi u_{\xi+1}) = 4$; $\varphi(w_\xi v_\xi) = 1$; $\varphi(v_\xi w_{\xi+1}) = 2$

Hence $\varphi$ is a total coloring of $L(\gamma_\eta)$ and therefore $\chi''(L(\gamma_\xi)) \le 5$. Since $\Delta(L(\gamma_\eta)) = \eta$ and by lemma 2.7, $\chi''(L(\gamma_\xi)) \ge \Delta(L(\gamma_\xi)) + 1 = 4 + 1 \ge 5$ and $\chi''(L(\gamma_\xi)) = 5$.

**Theorem 3.8:** let $M(\gamma_\eta)$ be the middle graph of lobster graph, then $\chi''(M(\gamma_\eta)) = 9$.

Proof: let $V(M(\gamma_\eta)) = \{v_\xi, u_\xi, u'_\xi, u''_\xi, u'''_\xi, w_\xi, w'_\xi, w''_\xi, w'''_\xi : 1 \le \xi \le \eta\} \bigcup \{x_\xi : 1 \le \xi \le \eta - 1\}$ and $E(M(\gamma_\eta)) = \{v_\xi u_\xi, u_\xi u'_\xi, u'_\xi u''_\xi, u''_\xi u'''_\xi, v_\xi w_\xi, w_\xi w'_\xi, w'_\xi w''_\xi, w''_\xi w'''_\xi, u_\xi u''_\xi, u_\xi w_\xi, w_\xi w''_\xi : 1 \le \xi \le \eta\}$ $\bigcup \{w_\xi x_\xi, x_\xi w_{\xi+1}, v_\xi x_\xi, x_\xi v_{\xi+1}, u_\xi x_\xi, x_\xi y_{\xi+1} : 1 \le \xi \le \eta - 1\}$ $\bigcup \{x_\xi x_{\xi+1} : 1 \le \xi \le \eta - 2\}$.

Define $\varphi : V(M(\gamma_\eta)) \bigcup E(M(\gamma_\eta)) \to \{1,2,...9\}$ as follows. The assigning of colors as noted below:

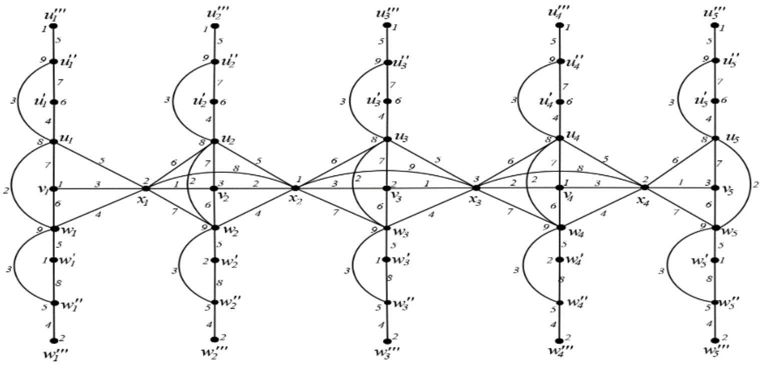

Fig 8: Total coloring of $M(\gamma_5)$

For $1 \le \xi \le \eta$

$\varphi(v_\xi) = 1,2,3;\ if\ \ \xi \equiv 1,2,0\ (\mathrm{mod}\,3)$ ; $\varphi(w'_\xi) = 1,2;\ if\ \ \xi \equiv 1,0\ (\mathrm{mod}\,2)$ ; $\varphi(w'''_\xi) = 2$
$\varphi(w''_\xi) = 5,6;\ if\ \ \xi \equiv 1,0\ (\mathrm{mod}\,2)$ ; $\varphi(u'''_\xi) = 1$; $\varphi(u''_\xi) = 9$ ; $\varphi(u_\xi) = 6$ ; $\varphi(u'_\xi) = 8$ ; $\varphi(w_\xi) = 9$ ;
$\varphi(v_\xi u_\xi) = 7$ ; $\varphi(u_\xi u'_\xi) = 4$ ; $\varphi(u'_\xi u''_\xi) = 7$ ; $\varphi(u''_\xi u'''_\xi) = 5$ ; $\varphi(v_\xi w_\xi) = 6$ ; $\varphi(w_\xi w'_\xi) = 5$ ; $\varphi(w'_\xi w''_\xi) = 8$

$\varphi(w''_\xi w'''_\xi) = 4$ ; $\varphi(u_\xi u''_\xi) = 3$ ; $\varphi(u_\xi w_\xi) = 2$ ; $\varphi(w_\xi w''_\xi) = 3$

For $1 \le \xi \le \eta - 1$

$\varphi(x_\xi) = 2,1,3;\ if\ \ \xi \equiv 1,2,0\ (\mathrm{mod}\,3)$ ; $\varphi(v_\xi x_\xi) = 3,2,4;\ if\ \ \xi \equiv 1,2,0\ (\mathrm{mod}\,3)$ ;
$\varphi(x_\xi v_{\xi+1}) = 1,3,2;\ if\ \ \xi \equiv 1,2,0\ (\mathrm{mod}\,3)$ ; $\varphi(w_\xi x_\xi) = 4$ ; $\varphi(x_\xi w_{\xi+1}) = 7$ ;

$\varphi(u_\xi x_\xi) = 5$ ; $\varphi(x_\xi u_{\xi+1}) = 6$

For $1 \le \xi \le \eta - 2$ , $\varphi(x_\xi x_{\xi+1}) = 8,9;\ if\ \ \xi \equiv 1,0\ (\mathrm{mod}\,2)$

Hence $\varphi$ is a total coloring of $M(\gamma_\eta)$ and therefore $\chi''(M(\gamma_\eta)) \le 9$ . Since $\Delta(M(\gamma_\eta)) = \eta$ and by lemma 2.7, $\chi''(M(\gamma_\eta)) \ge \Delta(M(\gamma_\eta)) + 1 = 8 + 1 \ge 9$ and $\chi''(M(\gamma_t)) = 9$.

**Theorem 3.9:** let $T(\gamma_\eta)$ be the middle graph of lobster graph, then $\chi''(T(\gamma_\eta)) = 9$.

Proof: let $V(T(\gamma_\eta)) = \{v_\xi, u_\xi, u'_\xi, u''_\xi, u'''_\xi, w_\xi, w'_\xi, w''_\xi, w'''_\xi : 1 \le \xi \le \eta\} \cup \{x_\xi : 1 \le \xi \le \eta - 1\}$ and

$E(T(\gamma_\eta)) = \{v_\xi u_\xi, u_\xi u'_\xi, u'_\xi u''_\xi, u''_\xi u'''_\xi, v_\xi w_\xi, w_\xi w'_\xi, w'_\xi w''_\xi, w''_\xi w'''_\xi, u_\xi u''_\xi, u'''_\xi u'_\xi, u'_\xi v_\xi, v_\xi w'_\xi, w'_\xi w'''_\xi,$
$u_\xi w_\xi, w_\xi w''_\xi : 1 \le \xi \le \eta\} \cup \{u_\xi x_\xi, v_\xi v_{\xi+1}, x_\xi u_{\xi+1}, w_\xi x_\xi, x_\xi w_{\xi+1}, v_\xi x_\xi, x_\xi v_{\xi+1} : 1 \le \xi \le \eta - 1\}$
$\cup \{x_\xi x_{\xi+1} : 1 \le \xi \le \eta - 2\}$ .

Define $\varphi : V(T(\gamma_\eta)) \cup E(T(\gamma_\eta)) \to \{1,2,...9\}$ as follows. The assigning of colors as noted below:

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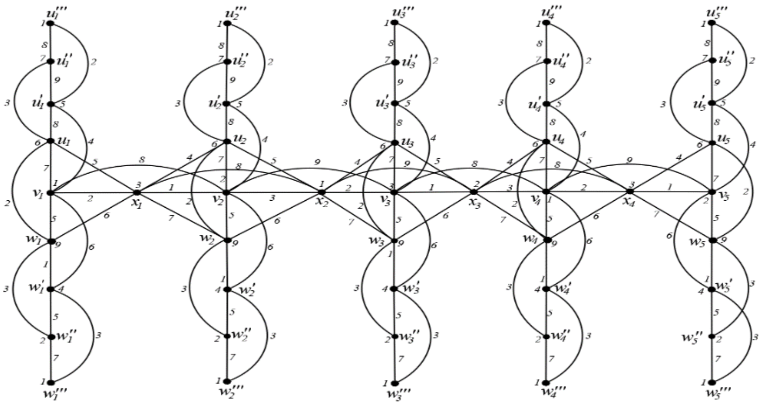

Fig 9: Total coloring of $T(\gamma_5)$

For $1 \le \xi \le \eta$

$\varphi(v_\xi) = 1, 2, 3;\ if\ \ \xi \equiv 1, 2, 0 \pmod 3$ ; $\varphi(u_\xi) = 6$ ; $\varphi(u_\xi^{'}) = 5$ ; $\varphi(u_\xi^{''}) = 7$ ; $\varphi(u_\xi^{'''}) = 1$ ; $\varphi(w_\xi) = 9$ ;

$\varphi(w_\xi^{'}) = 4$ ; $\varphi(w_\xi^{''}) = 2$ ; $\varphi(w_\xi^{'''}) = 1$ ; $\varphi(v_\xi u_\xi) = 7$ ; $\varphi(u_\xi u_\xi^{'}) = 8$ ; $\varphi(u_\xi^{'} u_\xi^{''}) = 9$ ; $\varphi(u_\xi^{''} u_\xi^{'''}) = 8$ ;

$\varphi(v_\xi w_\xi) = 5$ ; $\varphi(w_\xi w_\xi^{'}) = 1$ ; $\varphi(w_\xi^{'} w_\xi^{''}) = 5$ ; $\varphi(w_\xi^{''} w_\xi^{'''}) = 7$ ; $\varphi(u_\xi u_\xi^{''}) = 3$ ; $\varphi(u_\xi^{'''} u_\xi^{'}) = 2$ ;

$\varphi(u_\xi^{'} v_\xi) = 4$ ; $\varphi(v_\xi w_\xi^{'}) = 6$ ; $\varphi(w_\xi^{'} w_\xi^{'''}) = 3$ ; $\varphi(u_\xi w_\xi) = 2$ ; $\varphi(w_\xi w_\xi^{''}) = 3$

For $1 \le \xi \le \eta - 1$

$\varphi(x_\xi) = 3, 1, 2;\ if\ \ \xi \equiv 1, 2, 0 \pmod 3$ ; $\varphi(v_\xi x_\xi) = 2, 1, 3;\ if\ \ \xi \equiv 1, 2, 0 \pmod 3$ ;
$\varphi(x_\xi v_{\xi+1}) = 1, 2, 3;\ if\ \ \xi \equiv 1, 2, 0 \pmod 3$ ; $\varphi(w_\xi x_\xi) = 6$ ; $\varphi(x_\xi w_{\xi+1}) = 7$ ;

$\varphi(u_\xi x_\xi) = 5$ ; $\varphi(x_\xi u_{\xi+1}) = 4$ ; $\varphi(v_\xi v_{\xi+1}) = 8, 9;\ if\ \ \xi \equiv 1, 0 \pmod 2$

For $1 \le \xi \le \eta - 2$

$\varphi(x_\xi x_{\xi+1}) = 8, 9;\ if\ \ \xi \equiv 1, 0 \pmod 2$

Based on the above procedure, the graph $T(\gamma_\eta)$ is attained with 9 total colourable. Thus $\chi^{''}(T(\gamma_\eta)) \le 9$. Since $\Delta(T(\gamma_\eta)) = 8$ and by lemma 2.7, $\chi^{''}(T(\gamma_\eta)) \ge \Delta(T(\gamma_\eta)) + 1 = 8 + 1 \ge 9$ and Hence $\chi^{''}(T(\gamma_\eta)) = 9$.

## Conclusion

In this paper, the total chromatic number of triple star graph and its line, middle, total , splitting graph and lobster graph are obtained, the proofs provides an optimal solution to the total chromatic number for these graphs. Overall, delving into the total chromatic number of different graph classes and exploring the determination of total coloring in allocation across various families of graphs can contribute significantly to graph theory and related fields.

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