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COMPLEX PATTERNS IN FINANCIAL TIME SERIES THROUGH HIGUCHI'S FRACTAL DIMENSION

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Abstract

This paper analyzes the complexity of stock exchanges through fractal theory. Closing price indices of four stock exchanges with different industry sectors are selected. Degree of complexity is assessed through Higuchi's fractal dimension. Various window sizes are considered in evaluating the fractal dimension. It is inferred that the data considered as a whole represents random walk for all the four indices. Analysis of financial data through windowing procedure exhibits multi-fractality. Attempts to apply moving averages to reduce noise in the data revealed lower estimates of fractal dimension, which was verified using fractional Brownian motion. A change in the normalization factor in Higuchi's algorithm did improve the results. It is quintessential to focus on rural development to realize a standard and steady growth of economy. Tools must be devised to settle the issues in this regard. Micro level institutions are necessary for the economic growth of a country like India, which would induce a sporadic development in the present global economical scenario.

Keywords: Closing Price Indices; Fractals; Higuchi's Fractal Dimension; Complexity; Fractional Brownian Motion.

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1. INTRODUCTION

Fractal geometry developed in the last 20 years is one of the most scintillating and useful scientific discoveries of the century, owing its credit to Mandelbrot.[1] Fractals exhibit some similarity. Most of the fractals are self-similar i.e. magnification of any part resembles the original object in a specific manner. Fractals are those beyond the comprehension of Euclidean geometry. They are irregular. Recent years have witnessed the application of fractals in almost all fields. Our paper explores the application of fractals in finance from a different perspective.

It is a general notion that investing in companies which have grown up well over several years is safe. But this is not necessarily true in today's stock market. Taylor[2] and Peters[3] quote two terms, Random Walk Hypothesis (RWH) and Emerging Market Hypothesis (EMH). According to RWH, the prices in the market fluctuate in an erratic unpredictable manner. The EMH on the contrary states that recent changes are incorporated into the stock market. Mandelbrot[4] observed that price movements follow a family of distributions with high peaks and flat tails. Scale invariance property was also investigated. Cajueiro, Gojas and Tabak[5] analyzed long memory in financial data in the context of financial market liberalization. The results indicated strong long range dependence. It also highlights more efficient and more developed markets with long evidence of predictability. The analyzed Greek stock returns also possessed complex patterns.

It was also inferred by Cajueiro and Tabak[6] that herding behavior may be one of the causes for multifractality in the Japanese stock market. Analysis by Wang, Wu and Pan[7] has revealed that extreme events play an important role in the contributions of multi-fractality for US Dollar exchange rates. Paulch and Jackowska-Strumillo[8] have compared the efficiency of hybrid model versus hybrid model with fractal analysis for the short term forecast of Warsaw stock exchange. By hybrid model, they intend to mean a fusion of artificial neural networks and technical analysis. Results obtained by the latter method were more accurate.

Research contributions are pouring in since early times, undergoing an evolutionary exponential growth answering complicated questions that arise. These are few of the developments in the field of financial analysis. Utility of fractal dimension as a tool to assess the degree of complexity is also on the rise. We track, in particular the application of Higuchi's fractal dimension(HFD).

Paramanathan and Uthayakumar[9] have used HFD to analyze and measure the complexity of nonlinear electro encephalogram (EEG) signals. Bojic, Vuckovic and Kalauzi[10] have examined and compared the topographic distribution of HFD of EEG signals between wake and drowsy states. Bachmann, Lass, Suhhova and Hinrikus[11] have applied HFD for detection of extremely small hidden changes in human EEG.

Our initial probe on the literatures thus revealed two interesting facts about Higuchi's fractal dimension. First, recent years have witnessed its successful application in the study of EEG, an irregular time series. The second fact is its lack or minimal utility in the multi-fractal analysis of financial time series. This platform is the motivation for our paper.

Diversified perspective would aid in the analysis of stock exchange. This paper handles it in terms of irregularity indices, the fractal dimension. The space filling nature of financial data is analyzed through Higuchi's fractal dimension. To take a deeper insight at the fluctuations, various window sizes are used. The analysis of stock markets being Herculean, choice of data is crucial. A heterogeneous portfolio sounds a wise option at this juncture and so we have chosen the closing price indices of three stock exchanges-NASDAQ(NAS), Nikkei 225(NIKK), Dow Jones Industrial Average (DJI), with different industry sectors, along with Asia's first stock exchange, the Bombay Stock Exchange (BSE). Preprocessing of data is vital in any data analysis. This is to be emphatically followed in case of financial data, which is prone to high fluctuations. We have employed this first by ignoring no trading days and then normalizing the data. Normalization adds essence to the data set by removing noise, which are artifacts of fluctuation. Two normalization factors are attempted.

This paper is oriented as follows. Section 2 gives the rudiments of Brownian motion, fractional Brownian motion and Higuchi's fractal dimension. Changes in normalization factor using X_{norm} and X_{mov} are also presented. The results thus obtained are given in Sec. 3. Last section portrays the summary of the entire paper.

2. METHOD OF ANALYSIS

Fractal dimension is a measure of complexity. We use Higuchi's fractal dimension which is particularly suited for irregular time series. Normalization factor has been changed in Higuchi's formula in finding the length of the curve and its efficiency is tested using fractional Brownian motion signal. The notion of Brownian motion and fractional Brownian motion as in Falconer[12] is discussed below.

Let $X_\tau(t)$ denotes the position of the particle at time t. Then the positions $X_\tau(k\tau)$, $X_\tau((k+1)\tau)$ at times $k\tau$ and $(k+1)\tau$ respectively is equally likely to be $X_\tau(k\tau)+\delta$ or $X_\tau(k\tau)-\delta$. Assuming that the particle starts at time 0, then for $t>0$, the position at time 't' is described by the random variable

$$X_\tau(t) = \delta(Y_1 + \cdots + Y_{\lfloor t/\tau \rfloor}). \tag{1}$$

Considering the normalization factor $\delta = \sqrt{\tau}$

$$X_\tau(t) = \sqrt{\tau}(Y_1 + \cdots + Y_{\lfloor t/\tau \rfloor}). \tag{2}$$

For fixed t, if τ is small, then the distribution of the random variable $X_\tau(t)$ is approximately normal with mean 0 and variance t. Similarly, if t and h are fixed and τ is very small, then $X_\tau(t+h) - X_\tau(t)$ is approximately normal with mean 0 and variance h. Fractional Brownian motion of index α, $(0 < \alpha < 1)$ is defined to be Gaussian process $X : [0,\infty) \rightarrow \Re$ on some probability space such that

(i) $X\ (0) = 0$ and $X\ (t)$ is a continuous function of 't', with probability 1;
(ii) for all $t \geq 0$, $h > 0$, the increment $X\ (t + h) - X(t)$ is normally distributed with mean '0', variance $h^{2\alpha}$,

$$P([X(t+h) - X(t)] \leq x) = (2\pi)^{-1/2} h^{-\alpha} \int_{-\infty}^{x} \exp\left(\frac{-u^2}{2h^{2\alpha}}\right) du \tag{3}$$

Economic entities like share prices, stock exchange rates possess self- affine scaling. For $0 < \alpha < 1$, if $X(t)$ is the share price at time 't' then the increment $X\ (\gamma t) - X\ (\gamma t_0)$ has similar overall form to the increment $\gamma^\alpha(X(t) - X(t_0))$ with range $t > t_0$ and $\gamma > 0$. As Brownian motion is a most natural statistical self-affine model, it has been experimented in our paper. The relation between fractal dimension D and the index α is given by

$$D = 2 - \alpha. \tag{4}$$

For $\alpha = 1/2$, fractional Brownian motion given by Eq. (3) is reduced to Brownian motion. Iannaconne and Khokha[13] point out that when $\alpha = 1/2$, the data has no long range correlations i.e. the phenomenon is a random event. For $\alpha > 1/2$, the data is increasingly positively correlated and for $\alpha < 1/2$, it has negative correlations. Using Hurst's rescaled range analysis, a fractal time series can be characterized by the Hurst exponent H which relates to the fractal dimension of the time series by

$$D_H = 2 - H \tag{5}$$

for $0 \leq H \leq 1$. For fractional Brownian motion models, Eqs. (4) and (5) are equivalent. From Eq. (4), it follows that $D \quad = 1.5$ denotes a random walk, $1 < D < 1.5$, persistent nature and for $1.5 < D < 2$ denotes anti-persistent nature. If D varies with time then the process is said to be multifractal.

Table 1 **Higuchi's Fractal Dimension of Raw Data for all Indices.**

Index	D_{Hfd}
BSE	1.46
DJI	1.52
NAS	1.51
NIKK	1.50

MATLAB command 'wfbm' is used to invoke fractional Brownian motion, which is characterized by two parameters Hurst exponent H and the length of the signal N. The fractal dimension of the corresponding fractional Brownian motion is referred to as the 'theoretical D' (Table 2). The random process defined above may also be described on a finite interval $[t_1, t_2]$ in a similar manner. For the purpose of analysis, the period from 01/07/2004 to 20/11/2014, with varying total trading days, for all the four indices is considered. The required data was chosen from yahoo finance website. It was last accessed on 20th November 2014.

2.1. Higuchi's Fractal Dimension

Higuchi[14] designed a method for finding the fractal dimension of an irregular curve, which is given below.

Consider a finite set of time series observations taken at regular intervals.

$$X(1), X(2), X(3) \ldots X(N). \tag{6}$$

Table 2 Analysis of Higuchi's Fractal Dimension Values for all Four Stock Exchanges.

Index	Window 50		Window 150		Window 300	
	Max D_{Hfd}	Min D_{Hfd}	Max D_{Hfd}	Min D_{Hfd}	Max D_{Hfd}	Min D_{Hfd}
BSE	1.70	1.27	1.51	1.38	1.49	1.42
DJI	1.86	1.33	1.65	1.44	1.61	1.46
NAS	1.76	1.34	1.59	1.15	1.58	1.44
NIKK	1.81	1.29	1.61	1.40	1.56	1.47

From the given time series, the following new time series X_i^m is constructed:

$$X_i^m : X(m), X(m+i), X(m+2i)\dots$$
$$X\left(m + \lfloor\frac{N-m}{i}\rfloor i\right) \quad \text{with } m = 1, 2\dots i \tag{7}$$

Here $\lfloor\rfloor$ denotes Gauss' notation and both 'i', 'm' are integers. They represent the interval time and initial time, respectively. For a time interval 'i', a total of 'i' new sets of time series are obtained. The series is so constructed that there are no overlaps. The length of the curve X_i^m is defined as follows:

$$L_m(i) = \left\{\left(\sum_{l=1}^{\lfloor\frac{N-m}{i}\rfloor} |X(m+l\,i) - X(m+(l-1).i)|\right) * \frac{N-1}{\lfloor\frac{N-m}{i}\rfloor i}\right\} * \frac{1}{i} \tag{8}$$

The term $\frac{N-1}{\lfloor\frac{N-m}{i}\rfloor i}$ represents the normalization factor for the curve length of subset time series. Let $\langle L(i)\rangle$ be the average value over 'i' sets of $L_m(i)$. If

$$\langle L(i)\rangle \alpha i^{-D}, \tag{9}$$

then the curve is fractal with the dimension D. Using the method of least squares a straight line is fitted. The maximum value of 'i' is eight. We shall denote this Higuchi's fractal dimension as D_{Hfd}.

In order to observe the impact of normalization factor on the fractal dimension, a change was carried out in Higuchi's algorithm. Let X denote the actual data set. $X_{\max}$ denotes the maximum value in the series and $X_{\min}$ denotes the minimum value in the series. Then the normalized new series X_{norm} is obtained as follows:

$$X_{\mathrm{norm}} = \frac{X - X_{\min}}{X_{\max} - X_{\min}}. \tag{10}$$

Evaluation of fractal dimension is the same as above, which shall be denoted as D_{norm}. This idea was found useful for analyzing the closing prices in the context of neural networks by Patel and Marwala.[15]

The actual series X is replaced with the moving average series X_{mov}, as part of normalization. The period of moving average considered here is eleven years. In Higuchi's algorithm, Eqs. (8) and (9) therefore become

$$L_m(i) = \left(\sum_{l=1}^{\lfloor\frac{N-m}{i}\rfloor} |X_{\mathrm{mov}}(m+li) - X_{\mathrm{mov}}(m+(l-1).i)|\right) * \frac{1}{i}. \tag{11}$$

Fractal dimension is evaluated as before and is denoted by D_{mov}

$$\langle L(i)\rangle \alpha i^{-D_{\mathrm{mov}}}. \tag{12}$$

The above methods have been implemented using MATLAB.

3. RESULTS

Our analysis takes into account the four closing price indices BSE, DJI, NAS and NIKK. The number of data points varied from 2560 for NIKK stock exchange to 2618 for NAS stock exchange. This variation is due to the fact that nontrading days differ between countries. In order to have uniformity, the total number of data points was taken as 2560, the least of the values. The corresponding graph is presented in (Fig. 1). It is observed that D_{Hfd} value is 1.5, for all four indices, (for raw data) which implies a **random walk** (Table 1). Calculation of HFD is shown in (Fig. 2) for raw data of BSE sensex. Coefficient of x in $y = 1.46x + 12.9$ is Higuchi's fractal dimension.

To explore multi-fractality, D_{Hfd} was evaluated after dividing the data into various window sizes. A total of seven window sizes **50, 75, 100, 150, 200,**

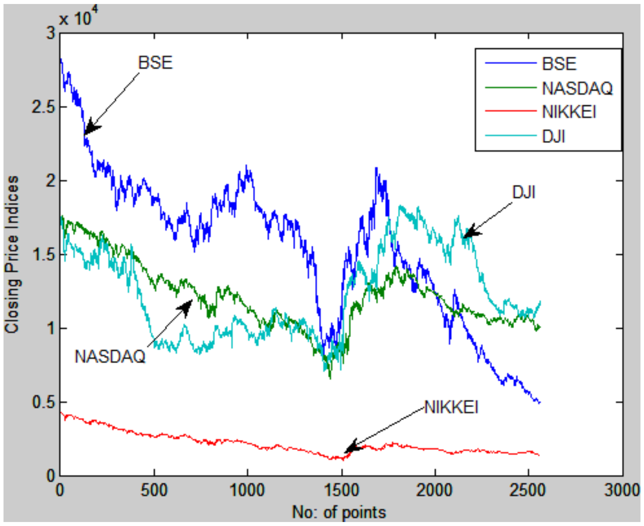


Fig. 1 Daily closing price indices of four stock exchanges.

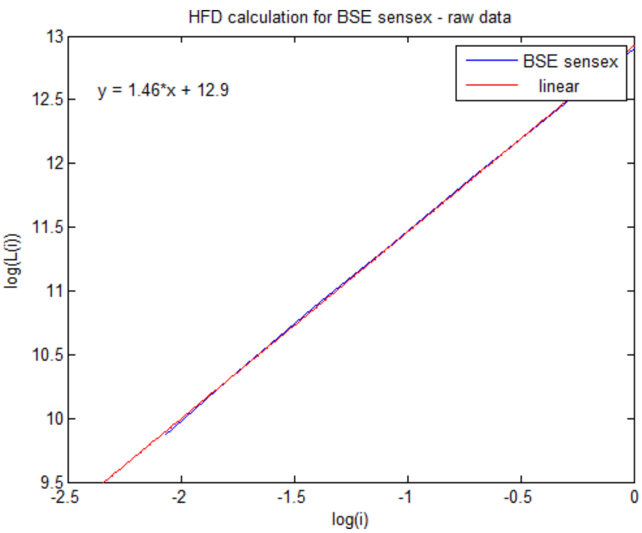


Fig. 2 log (i) versus $\log(L(i))$ for BSE sensex.

250, 300 were chosen. In contrary to the random walk exhibited earlier, we observe that the values extend on both sides of 1.5.

From the analysis of Higuchi's fractal dimension values presented in (Table 2), it is seen that D_{Hfd} values range from **1.27** to **1.67** in the case of BSE sensex for window size 50. Minimum value occurs for the financial period starting from "21st January 2014 to 2nd April 2014". This is the persistent behavior exhibited by the market and reflects monotonous tendency. It is an implication that the condition of market tends to be the same for the financial days following the above period. The extended period of roughly about two and a half months covering "3rd April 2014 to 18th June 2014" show a D_{Hfd} of 1.36 for the market trend. Beyond this period, the value gradually rises to 1.5 exhibiting random walk which reduces long term predictability. The maximum value occurs for the financial period starting from "16th August 2011 to 28th October 2011". This is a replica of the anti-persistent nature of market. Varying Higuchi's fractal dimension values for windows **50, 150, 300** are presented in (Fig. 3). This shows the presence of multi-fractality.

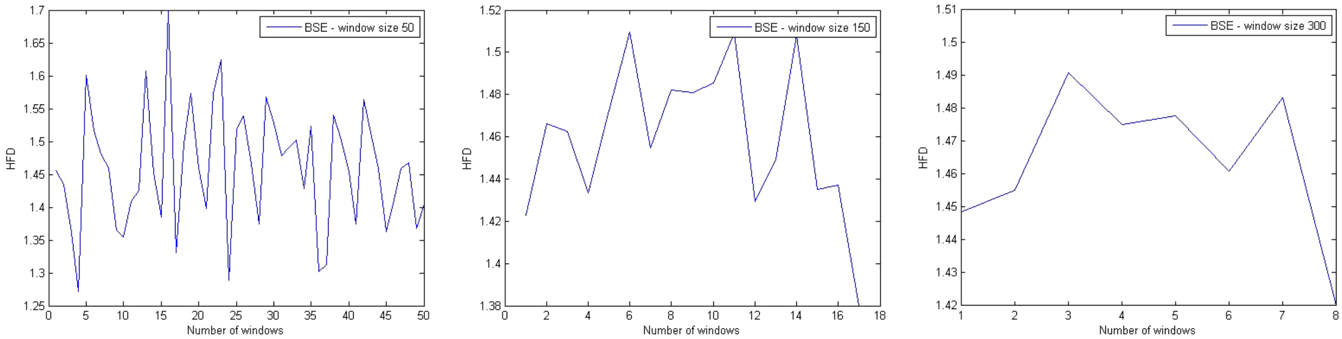
It is also interesting to note that a D_{Hfd} of **1.5** was observed in several cases even after windowing. The first of these value represents the business period between "18th December 2006 to 2nd March 2007", and the second between "29th February 2008" to "16th May 2008", representing a random walk. This is the 'crash period' Indian economy, when the market fell from 20,000 points to 8000 points. In view of window size 50, BSE sensex thus exhibits multi-fractality. Similar results were obtained for other indices too.

Table 3 **Higuchi's Fractal Dimension for Fractional Brownian Motion using X_{norm}.**

H	0.2	0.3	0.4	0.5	0.6
Theoretical D	1.8	1.7	1.6	1.5	1.4
Estimated D_H	1.75	1.7	1.59	1.5	1.42
Estimated D_{norm}	1.75	1.7	1.59	1.5	1.4

Noise is a matter of concern while dealing with any data set. We have attempted to reduce noise by using two different normalization procedures. Initially, X_{norm} has been used in place of X to evaluate D_{Hfd}. This was tested on fractional Brownian motion (fBm). Results of D_{Hfd} for fBm through both usual procedure and by considering X_{norm} are as in (Table 3). Satisfactory results prompted us to apply this for our data set. Only three window sizes 100, 200, 300 were considered for simplicity.

To analyze the impact of moving averages, the actual series was replaced by the 11 years moving averaged values. This was given as input to Higuchi's algorithm, removing normalization factor. Obviously, the new series contains 2550 points. Quite contrary to our expectations, very low estimates of fractal dimension were obtained. The results were also tested by retaining the normalization factor in Higuchi's algorithm. Low estimates were obtained here too, which was affirmed through fBm. This discouraging scenario prevented us from using moving average further and it was experimented only on window size 100. As exploration



(a) **(b)** **(c)**

Fig. 3 HFD of BSE sensex for (a) window size 50 (b) window size 150 (c) window size 300.

in this regard leads to deviation, we restrict at this point.

4. CONCLUSIONS

The analysis authenticates that financial data exhibit multi-fractality, which is evident from the varying values of D. The literature holds to its credit papers on both mono-fractal as well as multi-fractal cases. Our result complies with the second. The variation in results could be the end result of market calamity, during the specific period and the country concerned. This solely stands a testimony to the fact that financial data is highly volatile. It further justifies our attempt in using Higuchi's fractal dimension for financial analysis, which is tailor made for irregular and complex data. Measures taken in view of changing normalization factor have proved fruitful in the case of X_{norm} and futile in the case of X_{mov}. The venture of using X_{norm}, though has not improved the results to a greater significance, it has added special beauty to our work with more matching values of D_{Hfd} in certain cases. The results through X_{mov} can in no way decelerate the process of using moving average in financial analyzes, as it is an established tool for removal of noise and ours is only a particular case. In essence, our paper as a whole distinguishes itself from other papers in this line, primarily through the utility of D_{Hfd} in multi-fractal financial analysis and secondarily in change of normalization factor, paving way for more productive research with Higuchi's Fractal Dimension as a tool for multi-fractal analysis of complex data such as financial data. Our results in particular are but a reflection of Falconer's words: "Fractality is a crucial feature in mathematical modeling of finance".

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