Independent domination number in necklace graph

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Abstract

A set $S \subseteq V$ is a dominating set in $G(V,E)$ if every vertex in V not in S has atleast one neighbour in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A set S of vertices in a graph G is an independent dominating set of G if S is both an independent set and a dominating set of G. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set in G. In this paper, we obtain independent domination number for path necklace, cycle necklace and complete necklace networks. Further, we shown that equal domination number and independent domination number for P-necklace, C- necklace, K-necklace and star necklace graphs.

Keywords: Dominating Set, Independent Dominating Set, P-necklace, C- Necklace, K-Necklace, Star Necklace Graphs.

AMS Classification: 05C69, 05C76

Introduction

The domination in graphs is one of the concepts in graph theory which has attracted many researchers to work on it because it has the potential to solve many real life problems involving design and analysis of communication network as well as defense surveillance. Many variants of domination models are available in the existing literature such as independent domination, total domination, connected domination, edge domination and so on. In this paper, we solve independent domination number for chain silicate networks and cyclic silicate networks. Independent sets play an important role in graph theory and other area like discrete optimization.

The concept of an independent dominating set initiated in chessboard problems. In 1862 de Jaenisch posed the problem of finding the minimum number of mutually non-attacking queens that can be placed on a chessboard so that each square of the chessboard is attacked by at least one of the queens. A graph G may be formed from an 88 chessboard by taking the squares as the vertices with two vertices adjacent if a queen situated on one square attacks the other square. The graph G is known as the queens graph. The minimum number of mutually non-attacking queens that attack all the squares of a chessboard is the independent domination number $i(G)$. Independent dominating set also used in matching theory, coloring of graphs and in the theory of trees[1].

The theory of independent domination was formalized by Berge [2] and Ore in 1962 [3]. The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetniemi [4, 5]. Independent dominating sets in regular graphs and in cubic graphs in particular, are well studied in Goddard et al. [6], Kostichka [7] and Lam et al. [8]. Favaron [**9**] initiated the quest of finding sharp upper bounds for independent domination number in general graphs, as functions of n and δ. This work was extended by Haviland [10]. Cockayne et al. [11] obtained the upper bound for the product of the independent domination numbers of a graph and its complement while Shiu et al. [12] found the upper bounds for the independent domination number of triangle-free graphs and also characterized the extremal graphs achieving these upper bounds. Allan and Laskar [13] discussed the graphs with equal domination and independent domination numbers while Southey and Henning [14] considered the ratio of the independent domination number versus the domination number in a cubic graph and also characterized the graphs achieving this ratio of 4/3. Ao et al. [15] proved that for each $k \geq 4$, there exists a connected $k-$domination critical graph with independent domination number exceeding k.

Definition 1. [16] For $v \in V(G)$, the open neighbourhood of v, denoted as $N_G(v)$, is the set of vertices adjacent with v; and the closed neighbourhood of v, denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighbourhood of S is defined *as* $N_G(S) = \bigcup_{v \epsilon S} N_G(v)$ and theclosed neighbourhood of S is defined as$N_G[S] = N_G(S) \cup S$ For brevity, we denote $N_G(S)$by $N(S)$and $N_G[S]$by $N[S]$.

Definition 2. [16] For a graph $G(V, E)$, $S \subseteq V$ is a dominating set of G if every vertex in $V \setminus S$ has at least one neighbour in S. The domination number of G, denoted by γ(G), is the minimum cardinality of a dominating set of G.

Definition 3. [17] An independent set in a graph G is a set of pairwise non-adjacent vertices of G. A set S of vertices in a graph G is called an independent dominating set if S is both an independent set and a dominating set of G. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set in G.

Main Results

In this section, we prove that the domination number and independent domination number are equal for some special classes of graphs like P- necklace, C-necklace and K-necklace and star necklace graphs.

Lemma 1. Let G be a graph on n-vertices with cut vertices $v_1, v_2, \dots, v_m$, $m \geq 1$. Suppose for every cut vertex v_i of G, there exists components $H_{i1}, H_{i2}, \dots, H_{ik}$, $K_i \geq 1$, such that $G[(V(H_{ij}) \cup v_i]$ is a complete graph on r_{ij} vertices, $r_{ij} \geq 3$, $1 \leq j \leq k_i$, $1 \leq i \leq m$. Then $\gamma(G) \geq m$.

Proof. We claim that $|S| = m$. Let H_i, $1 \leq i \leq m$ be the components of $G \setminus v_i$such that$G[H_i \cup v_i] \cong Kti$. There are $m$vertex disjoint copies of Ktin G. Let S be the dominating set which contains $v_i s$ of every K_{ti}. Let H is isomorphic to the complete graph K_{ti}. Then the vertices of H is connected to through only v_i, $i = 1,2,\dots,m$. Let us assume the contrary that there exist a H such that $V(H) \cap S = \emptyset$. In both the cases of $V(H) \cap N(S) = \emptyset$ and $V(H) \cap N(S) \neq \emptyset$ the vertices $V(H_{ij}) \setminus v_i$ does not dominate any member of S, a contradiction.

P-Necklace

Definition 4. [18] Let $\mathrm{P_m}$ be a path on m vertices and let $\mathrm{K_{t_i\,i}}$ be complete graphs on $\mathrm{t_i}$ vertices, $1 \leq \mathrm{i} \leq \mathrm{m}$. The graph obtained by identifying any one vertex of $\mathrm{K_{t_i}}$ with $\mathrm{i^{th}}$ vertex of $\mathrm{P_m}$, $1 \leq \mathrm{i} \leq \mathrm{m}$, is called a P-necklace and is denoted by $\mathrm{PN(P_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})}$. See Figure 1.

The Algorithm given below computes the independent domination number in P- Necklace graph $\mathrm{PN(P_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})}$.

Independent Domination Algorithm in P- Necklace

Input: Let G be a P-necklace graph $PN(P_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$.

Algorithm: Label the vertices of $PN(P_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$ as follows:

(i) Label the vertices of the subgraph P_m of G as $v_1, v_2, \dots, v_m$ from left to right such that each complete graph K_{t_i} shares exactly one vertex of $v_i$of P_m, $1 \leq i \leq m$.

(ii) Let H_i, $1 \leq i \leq m$ be the components of $G \setminus v_i$ such that $G[H_i \cup v_i] \cong K_{t_i}$ and select exactly one vertex from each H_i in S.

Output: $\gamma(G) = i(G) = m$.

Proof of correctness: Let Sbe the set of $m$vertices chosen from each $H_i$of G. We

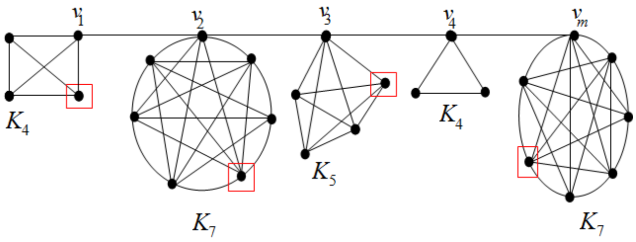

Figure-1: Red colored sqaured vertices constitutes the independent dominating set of P-Necklace $N(P_m, K)$

claim that, $|S| = m$. Let $v_1, v_2, \dots, v_m$ be the vertices of the subgraph P_m of G such that each complete graph K_{t_i}shares exactly one vertex of $v_i$of P_m, $1 \leq i \leq m$. It is easy to see that $v_i$is a cut vertex of G, $1 \leq i \leq m$. Let $H_i$be the components of $G \setminus v_i$such that $G[H_i \cup v_i] \cong K_{t_i}$. To prove each vertex in P-necklace is dominated, it is enough to prove that the vertices considered in Sdominates the each of $[H_i \cup v_i] \cong K_{t_i}$, $1 \leq i \leq m$. By Lemma 2.1, domination number of K_{t_i} is m. Therefore, $|S| = m$. Moreover, the vertices in $S$are pairwise non-adjacent. This implies that, $i(G) = m$.

Theorem 2. *Let G be a P-necklace graph $PN(P_m; K_{t_1}, K_{t_2}, \dots, K_{t_m})$, $n \geq 3$, $ti \geq 3$, $1 \leq i \leq m$ on $n = \sum_{i=1}^{m} t_i$ vertices. Then $\gamma(G) = i(G) = m$.*

C-Necklace

Definition 5. [2] *Let C_m be a cycle on m vertices and let K_{t_i}be complete graphs on$t_i$vertices, $1 \leq i \leq m$. The graph obtained by identifying any one vertex of $K_{t_i}$with i^{th} vertex of C_m, $1 \leq i \leq m$, is called a C-*

necklace and is denoted by $CN(C_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$. *See Figure* 2.

Independent Domination Algorithm in Algorithm H

Input: The C-necklace graph $CN(C_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$.

Algorithm: Label the vertices of $CN(C_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$as follows:

(i) Label the vertices of the subgraph C_m of G as $v_1, v_2, ..., v_m$ in the clock wise direction such that each complete graph K_{t_i} shares exactly one vertex of v_i of K_m , $1 \le i \le m$. (ii) Let H_i , $1 \le i \le m$ be the components of $G \backslash v_i$ such that $G[H_i \cup v_i] \cong K_{t_i}$ and select exactly one vertex from each H_i in S.

Output: $\gamma(G) = i(G) = m$.

Proof of correctness: Let S be the set of m vertices chosen from each H_i of G. We

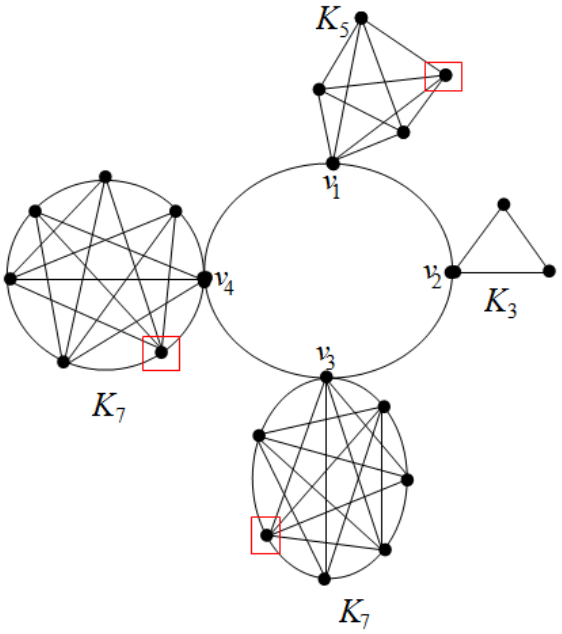

Figure-2: Red colored sqaured vertices constitutes the independent dominating set of C- Necklace $CN(C_4)$

claim that, $|S| = m$. Let $v_1, v_2, ..., v_m$ be the vertices of the subgraph C_m of G such that each complete graph K_{t_i} shares exactly one vertex of v_i of P_m , $1 \le i \le m$. It is easy to see that v_i is a cut vertex of G, $1 \le i \le m$. Let H_i be the components of $G \backslash v_i$ such that $G[H_i \cup v_i] \cong K_{t_i}$. To prove each vertex in C-necklace is dominated, it is enough to prove that the vertices considered in S dominates the each of $[H_i \cup v_i] \cong K_{t_i}$, $1 \le i \le m$. By Lemma 2.1, domination number of K_{t_i} is m. Therefore, $|S| = m$. Moreover, the vertices in S are pairwise non-adjacent. This implies that, $i(G) = m$.

Theorem 3. *Let* G *be a C-necklace graph* $CN(C_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$, $n \ge 3$, $ti \ge 3$, $1 \le i \le m$ *on* $n = \sum_{i=1}^{m} t_i$ *vertices. Then* $\gamma(G) = i(G) = m$.

K- Necklace

Definition 6. [18] *Let* K_m *and* K_{t_i} *be complete graphs on* m *and* t_i *vertices respectively,* $1 \le i \le m$. *The graph obtained by identifying any one vertex of* K_{t_i} *with the* i^{th} *vertex of* K_m, $1 \le i \le m$, *is called a* K *-necklace and is denoted by* $KN(K_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$. *See Figure* 3.

Independent Domination Algorithm in K-Necklace Graph

Input: The K-necklace $KN(K_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$.

Algorithm: Label the vertices of $KN(K_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$as follows:

(i) Label the vertices of the subgraph K_m of G as $v_1, v_2, ..., v_m$ in the clock wise direction such that each complete graph K_{t_i} shares exactly one vertex of v_i of K_m, $1 \le i \le m$.
(ii) Let H_i , $1 \le i \le m$ be the components of $G \backslash v_i$ such that $G[H_i \cup v_i] \cong K_{t_i}$, and select exactly one vertex from each H_i in S.
(iii) **Output**: $\gamma(G) = i(G) = m$.

Proof of Correctness: Let S be the set of m vertices chosen from each H_i of G. We claim that, $|S| = m$. Let $v_1, v_2, ..., v_m$ be the vertices of the subgraph K_m of G such that each complete graph K_{t_i} shares exactly one vertex of v_i of K_m, $1 \le i \le m$. It is easy to see that v_i is a cut vertex of G, $1 \le i \le m$. Let H_i be the components of $G \backslash v_i$ such that $G[H_i \cup v_i] \cong Kt_i$. To prove each vertex in K −necklace is dominated, it is enough to prove that the vertices considered in S dominates the each of $(H_i \cup v_i) \cong K_{t_i}$, $1 \le i \le m$. By Lemma 2.1, domination number of K_{t_i} is m. Therefore, $|S| = m$. Moreover, the vertices in S are pairwise non-adjacent. This implies that, $i(G) = m$.

Theorem 4. *Let* G *be the* $K-$ *necklace* $KN(K_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$, $m \ge 4$, $ti \ge 4$, $1 \le i \le m$. *Then* $\gamma(G) = i(G) = m$.

Star Necklace

Definition 7. [18] *Let* $K_{1,m}$ *be a star graph on* $m + 1$ *vertices and let* K_{t_i} *be a complete graphs on* t_i *vertices,* $1 \le i \le m$. *The graph obtained by identifying any one vertex of* K_{t_i} *with it* ^{h} *vertex of* $K_{1,m}$, $1 \le i \le m$, , *is called a star necklace and is denoted by* $N(K_{1,m}; K_{t_1}, K_{t_2}, ..., K_{t_m})$.*See Figure* 4.

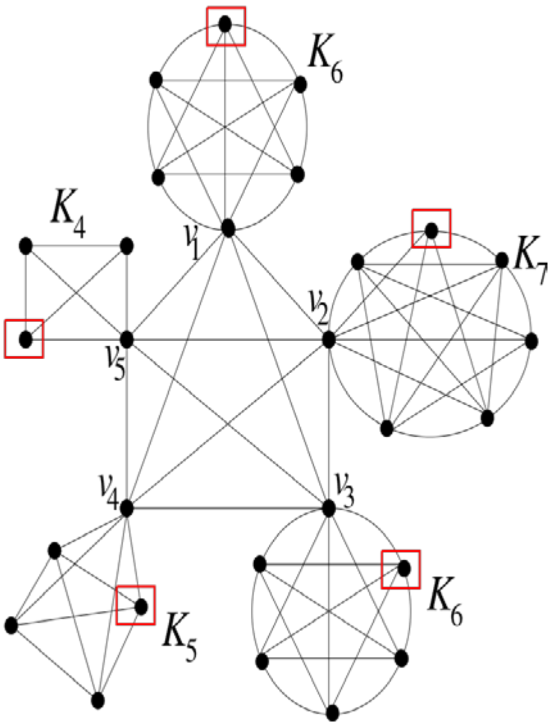

Figure-3: Red colored sqaured vertices constitutes the independent dominating set of *K*-Necklace $KN(K_5)$

Independent Domination Algorithm in Star Necklace

Input: Let G be a star necklace graph $KN\left(K_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m}\right)$.

Algorithm: Label the vertices of $KN(K_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$as follows:

(i) Label the root vertex of $K_{1,m}$ as v_0.

(ii) Label the vertices that are adjacent to root vertex v_0 consecutively $v_1, v_2, \ldots, v_m$ in the clock wise direction.

(iii) Let $H_i, 1 \le i \le m$ be the components of $G \backslash v_i s$ uch that that $G[Hi \cup vi] \cong Kt_i$. and select exactly one vertex from each $H_i in\ S$.

Output: $\gamma(G) = i(G) = m$.

Proof of correctness: Let S be the set of $m$vertices chosen from each H_i of G. We claim that, $|S| = m$. Let $v_1, v_2, \ldots, v_m$ be the vertices of the subgraph $K_{1,m}\, of\ G$ such that each complete graph Kti shares exactly one vertex of v_i of $K_{1,m}, 1 \le i \le m$. It is easy to see that $v_i$is a cut vertex of $G, 1 \le i \le m$. Let H_i be the components of $G \backslash v_i$ such that $G[H_i \cup v_i] \cong Kt_i$. To prove each vertex in star necklace is dominated, it is enough to prove that the vertices considered in S dominates the each of $(H_i \cup v_i) \cong K_{t_i}, 1 \le i \le m$. By Lemma 2.1, domination number of $K_{t_i} is\ m$. Therefore, $|S| = m$. Moreover, the vertices in $S$are pairwise non-adjacent. This implies that, $i(G) = m$.

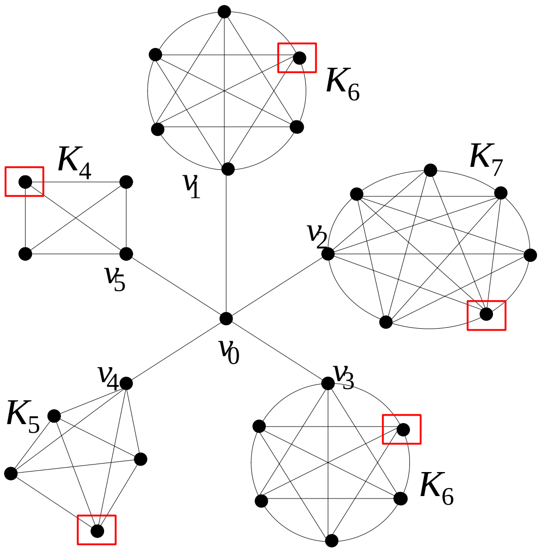

Figure-4: Red colored sqaured vertices constitutes the independent dominating set of Star Necklace $N(K1{,}5)$

Theorem 5. *Let G be the starne cklace* $KN(K_m; K_{t1}, K_{t2}, \ldots, K_{tm}), m \ge 4,\ ti \ge 4, 1 \le i \le m.$ *Then* $\gamma(G) = i(G) = m.$

Conclusion

In this paper, we obtained domination number and independent domination number for *P*-Necklace, *C*-Necklace, *K*-Necklace and Star Necklace. Further, we shown that domination number and independent domination number are equal for *P*-Necklace, *C*-Necklace, *K*-Necklace and Star Necklace.

References

1. Hedetniemi, S. T. and Laskar, R. C. (1991). Bibliography on domination in graphs and some basic definitions of domination parameters. *Annals of Discrete Mathematics*, 48: 257-277.
2. Berge, C. (1962). *Theory of Graphs and its Applications*. Methuen, London.
3. Ore, O. (1962). Theory of graphs. *Amer. Math. Soc. Transl.*, 38: 206-212.
4. Cockayne, E.J. and Hedetniemi, S.T. (1974). Independent graphs. *Congr. Numer.*, 10: 471-491.
5. Cockayne, E.J. and Hedetniemi, S.T. (1977). Towards a theory of domination in graphs. *Networks*, 7: 247-261.
6. Goddard, W., Henning, M. A., Lyle, J., and Southey, J. (2012). On the independent domination number of regular graphs. *Annals of Combinatorics*, 16(4): 719-732.
7. Kostochka, A. V. (1993). The independent domination number of a cubic 3-connected graph can be much larger than its domination number. *Graphs and Combinatorics*, 9(2-4): 235-237.
8. Lam, P. C. B., Shiu, W. C., and Sun, L. (1999). On independent domination number of regular graphs. *Discrete Mathematics*, 202(1-3): 135-144.
9. Favaron, O. (1988). Two relations between the parameters of independence and irredundance. *Discrete Mathematics*, 70(1): 17-20.
10. Haviland, J. (2007). Upper bounds for independent domination in regular graphs. *Discrete Mathematics*, 307(21): 2643-2646.
11. Cockayne, E. J., Favaron, O., Li, H., and MacGillivray, G. (1991). The product of the independent domination

numbers of a graph and its complement. *Discrete mathematics*, 90(3): 313-317.
12. Shiu, W. C., Chen, X. G., and Chan, W. H. (2010). Triangle-free graphs with large independent domination number. *Discrete Optimization*, 7(1-2), 86-92.
13. Allan, R. B., and Laskar, R. (1978). On domination and independent domination numbers of a graph. *Discrete mathematics*, 23(2): 73-76.
14. Southey, J., and Henning, M. A. (2013). Domination versus independent domination in cubic graphs. *Discrete Mathematics*, 313(11): 1212-1220.
15. Ao, S., Cockayne, E. J., MacGillivray, G., and Mynhardt, C. M. (1996). Domination critical graphs with higher independent domination numbers. *Journal of Graph Theory*, 22(1): 9-14.
16. Haynes, T. W., Hedetniemi, S. M., Hedetniemi, S. T., and Henning, M. A. (2002). Domination in graphs applied to electric power networks. *SIAM Journal on Discrete Mathematics*, 15(4): 519-529.
17. Vaidya, S. K., and Pandit, R. M. (2016). Independent Domination in Some Wheel Related Graphs. *Applications and Applied Mathematics*, 11: 397-407.
18. Rajan, R. S., Parthiban, N., and Rajalaxmi, T. M. (2015). Embedding of Recursive Circulants into Certain Necklace Graphs. *Mathematics in Computer Science*, 9(2): 253-263.