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ON TWIN EDGE COLORING OF SOME LADDER GRAPHS, LINE GRAPH OF COMB GRAPH AND TWIG GRAPH

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Abstract. A twin k-edge coloring of a graph G is a proper edge coloring of G that requires colors from Z_k and that induced a proper vertex coloring on G, where the color of a vertex v is the sum in Z_k of the colors of the edges incident with v. The least k for which G has a twin edge k-coloring is known as twin chromatic index of G and it is denoted by $\chi_t{}'(G)$. In this paper, we obtained the twin chromatic index of ladder graph, triangular ladder graph, slanting ladder graph, line graph of comb graph and line graph of twig graph.

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Key words and phrases : Ladder graph, line graph, comb graph, twig graph, twin edge coloring and twin chromatic index.

1. Introduction

Let $G(V, E)$ be a simple graph. A proper vertex coloring of a graph is an assignment of colors to the vertices of a graph such that no two adjacent vertices share the same color. The chromatic number of G, denoted as $\chi(G)$ is the least k-colors required for an appropriate vertex coloring. Each edge in G is assigned a color from a set of k-colors such that no two adjacent edges share the same color. The chromatic index of G, represented by $\chi'(G)$, is the minimum k- colors suffice for proper edge coloring.

In Chartrand [5], a related coloring was promptly introduced and also studied [2] and [3]. The above specific topic was explored in this paper. The twin edge coloring is defined as the following: let $c : E(G) \to Z_k$ be a proper edge $k-$coloring of G for some integer $k \geq 2$. A vertex $k-$coloring $c' : E(G) \to Z_k$ is

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defined by

$$c'(u) = \sum_{e \in E_u} c(e)$$

in Z_k, where E_u is the set of edges of G incident with a vertex u and the indicated sum is computed in Z_k. If the induced vertex $k-$ coloring $c'(u)$ is proper, then c is called a twin edge $k-$ coloring of G. The minimum k for which G has a twin edge $k-$ coloring is called the twin chromatic index of G and it is denoted by ${\chi_t}'(G)$. As a twin edge coloring is a proper edge coloring of G as well as its induces a proper vertex coloring of G, and it follows that

$${\chi_t}'(G) \geq max\{\chi(G), \chi'(G)\}.$$

In [2], the twin chromatic index of path, complete and complete bipartite graphs are presented. In [3], Andrews present results for trees, grids, prisms and permutation graphs of 5 cycles with $\Delta = 6$ and also verified the conjecture. Furthermore, in [4], the results have been verified for different types of tree graphs such as brooms, double star and some regular trees.

Anantharaman [1] determines the twin chromatic indices of total graph of path and cycles and also established twin chromatic indices of some special graphs are $\Delta + 2$. In [8], provides the twin chromatic indices of wheel graphs, Tolentino et.al [12]investigates the twin chromatic indices of some graphs with $\Delta = 3$ namely circulant graphs $C_n(1, \frac{n}{2})$, where $n \geq 8$ and $n \equiv 0(mod\ 4)$ and some generalized Peterson graphs. In [9], calculates the twin chromatic index of splitting, middle, shadow graph and direct product of path and cycle. In [11], presents the results in twin chromatic index of the square graphs and product graphs. If $n \geq 4$, then ${\chi_t}'(P_4^2) = 4, {\chi_t}'(P_5^2) = 4$, and for $n \geq 6$, ${\chi_t}'(P_n^2) = 5$. If $n \geq 6$ and $n \neq 7$ then ${\chi_t}'(C_n^2) = 5$, and ${\chi_t}'(C_7^2) = \Delta(C_7^2) + 2 = 6$. If $m \geq 3$ and $n \geq 3$, then the twin chromatic index of cartesian product of cycle with path is 5. In [14] determines the twin chromatic index of infinite square, hexagonal and triangular lattices. Tolentino et.al [13] determines on twin edge coloring in m-ary trees and also identifies upper bound for the twin chromatic index of all trees of order at least 3. Johnston et al. [7] obtained an upper bound for the twin chromatic index of a graph. Naveen [10] investigated the twin edge coloring of the alternate triangular snake, Triangular snake and double alternate triangular snake.

Conjecture 1.1[3]: If G is a connected graph of order at least 3 that is not a $5-$ cycle, then $\chi_t^{'}(G) \leq \Delta(G) + 2$.

Observation 1.2[3]: If G is a connected graph, contains two adjacent vertices of degree $\Delta(G)$, then $\chi_t^{'}(G) \geq \Delta(G) + 1$.

2. Preliminaries

Definition 2.1. The *Ladder graph* L_n formed by taking two parallel paths of length n and connecting corresponding vertices with additional edges.

Definition 2.2. The *Triangular ladder* TL_n, $n \geq 2$ is the attachment of edges between certain pairs of vertices on the ladder. Specifically, attaching the edges $u_i v_{i+1}$ between the vertices u_i and v_{i+1} for each $i \in \{1, 2, ..., n-1\}$.

Definition 2.3. The *Slanting ladder* SL_n is a graph constructed by taking two copies of path P_n then forming edges by connecting u_{i+1} and v_i, for all $i \in \{1, 2, ..., n-1\}$.

Definition 2.4. The *Comb graph* P_n^+ [6] is the graph obtained from a path by attaching one pendent edge at each vertex of a path.

Definition 2.5. The *Twig graph* T_n [6] is a tree obtained from a path by attaching exactly two pendant edges to each internal vertices of the path.

In the subsequent section, the twin edge chromatic index of ladder L_n, triangular ladder TL_n, slanting ladder SL_n, line graph of comb graph $L(P_n^+)$, and line graph of twig graph $L(T_n)$ are obtained.

3. Main results

Theorem 3.1. *Let* L_n *be the ladder with* $n \geq 3$, *then* $\chi_t{}'(L_n) = 4$.

Proof. Consider L_n. For $\xi \in \{1, 2, \ldots, n\}$, let $V(L_n) = \{u_\xi, v_\xi\}$, for $\xi \in \{1, 2, \ldots, n-1\}$, let the edges be $r_\xi = u_\xi u_{\xi+1}, r_\xi^{'} = v_\xi v_{\xi+1}$ and for $\xi \in \{1, 2, \ldots, n\}$, let $r_\xi^{''} = u_\xi v_\xi$.
For $n = 3$, define $c : E(L_3) \to Z_4$ as follows: $c(r_1) = 0, c(r_2) = 1, c(r_1^{'}) = 3, c(r_2^{'}) = 0, c(r_1^{''}) = 1$, $c(r_2^{''}) = 2, c(r_3^{''}) = 3$. The induced vertex coloring is: $c'(u_1) = 1, c'(u_2) = 3, c'(u_3) = 0, c'(v_1) = 0, c'(v_2) = 1, c'(v_3) = 3$ and it is proper. Hence $\chi_t^{'}(L_3) \leq 4$. By observation 1.2, $\chi_t^{'}(L_3) \geq 4$ and so $\chi_t^{'}(L_3) = 4$.
For $n \geq 4$, define $c : E(L_n) \to Z_4$ as follows:
For $\xi \in \{1, 2, \ldots, n-1\}$

$$c(r_\xi) \equiv \begin{cases} 0, & \text{if } \xi \equiv\ 1\ (mod\ 4) \\ 1, & \text{if } \xi \equiv\ 2\ (mod\ 4) \\ 2, & \text{if } \xi \equiv\ 3\ (mod\ 4) \\ 3, & \text{if } \xi \equiv\ 0\ (mod\ 4) \end{cases}$$

$$c(r_\xi^{'}) \equiv \begin{cases} 1, & \text{if } \xi \equiv\ 1\ (mod\ 4) \\ 2, & \text{if } \xi \equiv\ 2\ (mod\ 4) \\ 3, & \text{if } \xi \equiv\ 3\ (mod\ 4) \\ 0, & \text{if } \xi \equiv\ 0\ (mod\ 4) \end{cases}$$

$c(r_{n-1}^{'}) = 3$, if $n \equiv\ 3\ (mod\ 4)$

$$c(r_\xi^{''}) \equiv \begin{cases} 2, & \text{if } \xi \equiv\ 1\ (mod\ 4) \\ 3, & \text{if } \xi \equiv\ 2\ (mod\ 4) \\ 0, & \text{if } \xi \equiv\ 3\ (mod\ 4) \\ 1, & \text{if } \xi \equiv\ 0\ (mod\ 4) \end{cases}$$

$c(r^{''}_{n-1}) = 2$, if $n \equiv\ 3\ (mod\ 4)$; $c(r^{''}_{n-1}) = 0$, if $n \equiv\ 0\ (mod\ 4)$; $c(r^{''}_{n}) = 0$, if $n \equiv\ 0\ (mod\ 4)$
The induced vertex coloring is:
$c'(u_1) = c(r_1) + c(r^{''}_1) = 0 + 2 \equiv 2$; $c'(v_1) = c(r^{'}_1) + c(r^{''}_1) = 1 + 2 \equiv 3$
For $\xi \in \{2, 3, \dots, n-1\}$

$$c'(u_\xi) = c(r_{\xi-1}) + c(r_\xi) + c(r^{''}_\xi) \equiv \begin{cases} 0, & \text{if } \xi \equiv\ 2\ (mod\ 4) \\ 3, & \text{if } \xi \equiv\ 3\ (mod\ 4) \\ 2, & \text{if } \xi \equiv\ 0\ (mod\ 4) \\ 1, & \text{if } \xi \equiv\ 1\ (mod\ 4) \end{cases}$$

$$c'(u_n) \equiv \begin{cases} 2, & \text{if } n \equiv\ 0\ (mod\ 4) \\ 1, & \text{if } n \equiv\ 1, 3\ (mod\ 4) \\ 3, & \text{if } n \equiv\ 2\ (mod\ 4) \end{cases}$$

$$c'(v_\xi) = c(r^{'}_{\xi-1}) + c(r^{'}_\xi) + c(r^{''}_\xi) \equiv \begin{cases} 2, & \text{if } \xi \equiv\ 2\ (mod\ 4) \\ 1, & \text{if } \xi \equiv\ 3\ (mod\ 4) \\ 0, & \text{if } \xi \equiv\ 0\ (mod\ 4) \\ 3, & \text{if } \xi \equiv\ 1\ (mod\ 4) \end{cases}$$

$$c'(v_n) \equiv \begin{cases} 3, & \text{if } n \equiv\ 0, 3\ (mod\ 4) \\ 2, & \text{if } n \equiv\ 1\ (mod\ 4) \\ 0, & \text{if } n \equiv\ 2\ (mod\ 4) \end{cases}$$

Hence c is a twin edge $4-$coloring of L_n and therefore $\chi^{'}_t(L_n) \leq 4$. By observation 1.2. This implies that $\chi^{'}_t(L_n) \geq 4$, and so $\chi^{'}_t(L_n) = 4$. □

Example 3.2. The graph L_7 and its twin edge coloring is shown in Fig. 1.

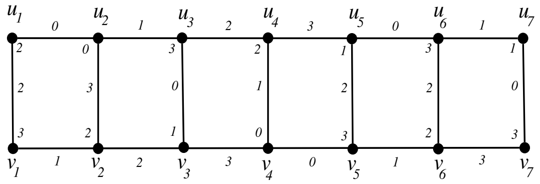

FIGURE 1. L_7 and its twin edge coloring

The edges $c(r_1), c(r_2), \dots, c(r_6)$ is of the form $0, 1, 2, 3, 0, 1$; the edges $c(r_1'), c(r_2'), \dots, c(r_6')$ is of the form $1, 2, 3, 0, 1, 3$; $c(r_1''), c(r_2''), \dots,\ c(r^{''}_7) = 2, 3, 0, 1, 2, 2, 0$, the induced vertex coloring, the sequence of vertices $c'(u_1), c'(u_2), \dots, c'(u_7)$ is of the form $2, 0, 3, 2, 1, 3, 1$; the vertices $c'(v_1), c'(v_2), \dots, c'(v_7)$ is of the form $3, 2, 1, 0, 3, 2, 3$ and it is proper. Hence c is a twin edge $4-$coloring of L_7 and therefore $\chi_t'(L_7) \leq 4$. By observation 1.2, $\chi_t'(L_7) \geq 4$, and so $\chi_t'(L_7) = 4$.

Example 3.3. The graph L_8 and its twin edge coloring is shown in Fig. 2.

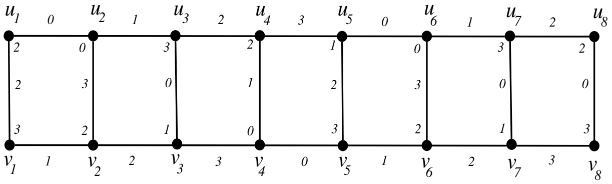

FIGURE 2. L_8 and its twin edge coloring

Theorem 3.4. *Let* TL_n, *be the triangular ladder with* $n \geq 3$, *then* $\chi_t{}'(TL_n) = 5$.

Proof. Consider TL_n. For $\xi \in \{1, 2, \ldots, n\}$, let $V(TL_n) = \{u_\xi, v_\xi\}$, for $\xi \in \{1, 2, \ldots, n-1\}$, let the edges be $r_\xi = u_\xi u_{\xi+1}, r'_\xi = v_\xi v_{\xi+1}, r''_\xi = u_\xi v_{\xi+1}$ and for $\xi \in \{1, 2, \ldots, n\}$, let $r'''_\xi = u_\xi v_\xi$.
For $n = 3$, define $c : E(TL_3) \to Z_5$ as follows:
$c(r_1) = 1, c(r_2) = 0, c(r'_1) = 1, c(r'_2) = 0, c(r'''_1) = 0,$
$c(r'''_2) = c(r'''_3) = 3, c(r''_1) = 4, c(r''_2) = 2.$
The induced vertex coloring is:
$c'(u_1) = 0$, $c'(u_2) = 1$, $c'(u_3) = 3$; $c'(v_1) = 1$; $c'(v_2) = 3$; $c'(v_3) = 0$ and it is proper. Hence $\chi_t{}'(TL_3) \leq 5$. By observation 1.2, $\chi_t{}'(TL_3) \geq 5$ and so $\chi_t{}'(TL_3) = 5$.
For $n \geq 4$, define $c : E(TL_n) \to Z_5$ as follows.
For $\xi \in \{1, 2, \ldots, n-1\}$,

$$c(r_\xi) = c(r'_\xi) \begin{cases} 1, & \text{if } \xi \equiv \ 1 \ (mod\ 2) \\ 0, & \text{if } \xi \equiv \ 0 \ (mod\ 2) \end{cases}$$

$c(r'''_1) = 0,$
For $\xi \in \{2, 3, \ldots, n-1\}$,

$$c(r''_\xi) = \begin{cases} 4, & \text{if } \xi \equiv \ 1 \ (mod\ 3) \\ 2, & \text{if } \xi \equiv \ 2 \ (mod\ 3) \\ 3, & \text{if } \xi \equiv \ 3 \ (mod\ 3) \end{cases}$$

$$c(r'''_\xi) = \begin{cases} 0, & \text{if } \xi = 0 \\ 3, & \text{if } \xi \equiv \ 2 \ (mod\ 3) \\ 4, & \text{if } \xi \equiv \ 0 \ (mod\ 3) \\ 2, & \text{if } \xi \equiv \ 1 \ (mod\ 3) \end{cases}$$

$$c(r'''_n) = \begin{cases} 0, & \text{if } n \equiv \ 4 \ (mod\ 6) \\ 1, & \text{if } n \equiv \ 5 \ (mod\ 6) \\ 4, & \text{if } n \equiv \ 0 \ (mod\ 6) \\ 2, & \text{if } n \equiv \ 1 \ (mod\ 6) \\ 3, & \text{if } n \equiv \ 2, 3 \ (mod\ 6) \end{cases}$$

The induced vertex coloring is:

$c'(u_1) = c(r_1) + c(r_1^{'''}) + c(r_1^{''}) = 1 + 0 + 4 = 5 \equiv 0$
For $\xi \in \{2, 3, \ldots, n-1\}$

$$c'(u_\xi) = c(r_{\xi-1}) + c(r_\xi) + c(r_\xi^{'''}) + c(r_\xi^{''}) \equiv \begin{cases} 1, & \text{if } \xi \equiv\ 2\ (mod\ 3) \\ 3, & \text{if } \xi \equiv\ 0\ (mod\ 3) \\ 2, & \text{if } \xi \equiv\ 1\ (mod\ 3) \end{cases}$$

$$c'(u_n) = c(r_{n-1}) + c(r_n^{''}) \equiv \begin{cases} 1, & \text{if } n \equiv\ 4,5\ (mod\ 6) \\ 0, & \text{if } n \equiv\ 0\ (mod\ 6) \\ 2, & \text{if } n \equiv\ 1\ (mod\ 6) \\ 4, & \text{if } n \equiv\ 2\ (mod\ 6) \\ 3, & \text{if } n \equiv\ 3\ (mod\ 6) \end{cases}$$

$c'(v_1) = c(r_1^{'''}) + c(r_1^{'}) = 0 + 1 = 1 \equiv 1$

$$c'(v_\xi) = c(r_{\xi-1}^{'}) + c(r_\xi^{'}) + c(r_{\xi-1}^{'''}) + c(r_{\xi-1}^{''}) \equiv \begin{cases} 3, & \text{if } \xi \equiv\ 2\ (mod\ 3) \\ 2, & \text{if } \xi \equiv\ 0\ (mod\ 3) \\ 1, & \text{if } \xi \equiv\ 1\ (mod\ 3) \end{cases}$$

$$c'(v_n) = c(r_{n-1}^{'}) + c(r_n^{'''}) \equiv \begin{cases} 4, & \text{if } n \equiv\ 4\ (mod\ 6) \\ 0, & \text{if } n \equiv\ 1,3,5\ (mod\ 6) \\ 2, & \text{if } n \equiv\ 0\ (mod\ 6) \\ 3, & \text{if } n \equiv\ 2\ (mod\ 6) \end{cases}$$

and it is proper. Hence c is a twin edge 5−coloring of TL_n and therefore $\chi_t{}'(TL_n) \leq 5$. By observation 1.2. This implies that $\chi_t{}'(TL_n) \geq 5$, and so $\chi_t{}'(TL_n) = 5$. □

Example 3.5. The graph TL_6 and its twin edge coloring is shown in Fig. 3.

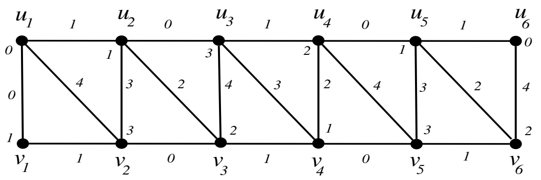

FIGURE 3. TL_6 and its twin edge coloring

Example 3.6. The graph TL_7 and its twin edge coloring is shown in Fig. 4.

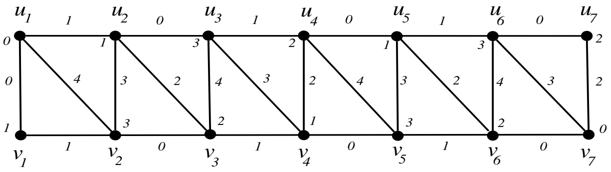

FIGURE 4. TL_7 and its twin edge coloring

Example 3.7. The graph TL_8 and its twin edge coloring is shown in Fig. 5.

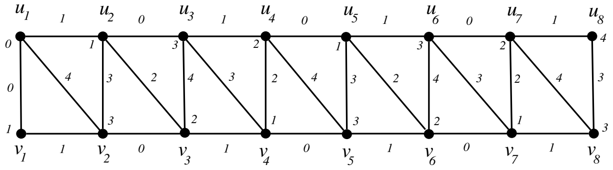

FIGURE 5. TL_8 and its twin edge coloring

Example 3.8. The graph TL_9 and its twin edge coloring is shown in Fig. 6.

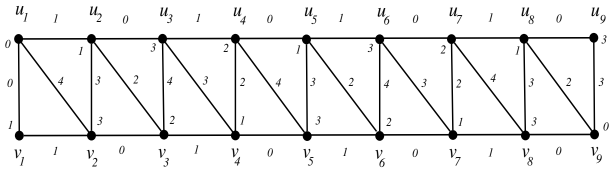

FIGURE 6. TL_9 and its twin edge coloring

Example 3.9. The graph TL_{10} and its twin edge coloring is shown in Fig. 7.

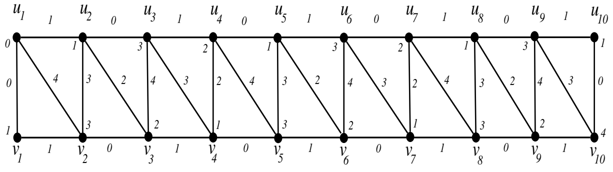

FIGURE 7. TL_{10} and its twin edge coloring

Example 3.10. The graph TL_{11} and its twin edge coloring is shown in Fig. 8.

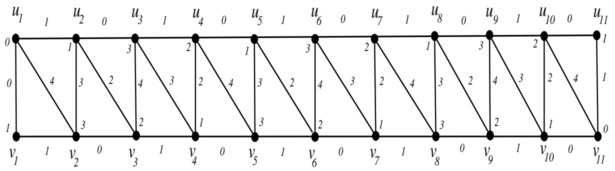

FIGURE 8. TL_{11} and its twin edge coloring

Theorem 3.11. *Let SL_n, be the slanting ladder with $n \geq 3$, then $\chi_t{}'(SL_n) = 4$.*

Proof. Consider SL_n. For $\xi \in \{1, 2, \ldots, n\}$,
let $V(TL_n) = \{u_\xi, v_\xi\}$, for $\xi \in \{1, 2, \ldots, n-1\}$,
let the edges $r_\xi = u_\xi u_{\xi+1}, r_\xi^{'} = v_\xi v_{\xi+1}, r_\xi^{''} = u_{\xi+1} v_\xi$.
For $n = 3$, define $c : E(SL_3) \to Z_3$ as follows:
$c(r_1) = 1, c(r_2) = 2, c(r_1^{'}) = 2, c(r_2^{'}) = 1;$
$c(r_1^{''}) = c(r_2^{''}) = 0;$
The induced vertex coloring is: $c'(u_1) = 1$, $c'(u_2) = 0$, $c'(u_3) = 2$, $c'(v_1) = 2$; $c'(v_2) = 0$, $c'(v_3) = 1$, and it is proper. Hence $\chi_t{}'(SL_3) \leq 3$. By observation 1.2, $\chi_t{}'(SL_3) \geq 3$ and so $\chi_t{}'(SL_3) = 3$.
For $n \geq 4$, define $c : E(SL_n) \to Z_4$ as follows:

$$c(r_\xi) = c(r_\xi^{'}) \equiv \begin{cases} 1, & \text{if } \xi \equiv \ 1 \ (mod \ 3) \\ 2, & \text{if } \xi \equiv \ 2 \ (mod \ 3) \\ 3, & \text{if } \xi \equiv \ 3 \ (mod \ 3) \\ 0, & \text{if } \xi \equiv \ 0 \ (mod \ 3) \end{cases}$$

$c(r_{n-1}) = 3, \ n \equiv 3 (mod 4)$
For $\xi \in \{1, 2, \ldots, n-1\}$

$$c(r_\xi^{''}) \equiv \begin{cases} 3, & \text{if } \xi \equiv \ 1 \ (mod \ 3) \\ 0, & \text{if } \xi \equiv \ 2 \ (mod \ 3) \\ 1, & \text{if } \xi \equiv \ 3 \ (mod \ 3) \\ 2, & \text{if } \xi \equiv \ 0 \ (mod \ 3) \end{cases}$$

The induced vertex coloring is:

$$c'(u_1) = \begin{cases} 1, & \text{if } n \equiv \ 0, 1, 3 \ (mod \ 4) \\ 3, & \text{if } n \equiv \ 2 \ (mod \ 4) \end{cases}$$

For $\xi \in \{2, 3, \ldots, n-2\}$

$$c'(u_\xi) = c(r_{\xi-1}) + c(r_\xi) + c(r_{\xi-1}^{''}) \equiv \begin{cases} 3, & \text{if } \xi \equiv \ 1 \ (mod \ 4) \\ 2, & \text{if } \xi \equiv \ 2 \ (mod \ 4) \\ 1, & \text{if } \xi \equiv \ 3 \ (mod \ 4) \\ 0, & \text{if } \xi \equiv \ 0 \ (mod \ 4) \end{cases}$$

$$c'(u_{n-1}) = c(r_{n-2}) + c(r_{n-1}) + c(r^{''}_{n-2}) \equiv \begin{cases} 1, & \text{if } n \equiv\ 0\ (mod\ 4) \\ 0, & \text{if } n \equiv\ 1, 2\ (mod\ 4) \\ 2, & \text{if } n \equiv\ 3\ (mod\ 4) \end{cases}$$

$$c'(u_n) = c(r_{n-1}) + c(r^{''}_{n-1}) \equiv \begin{cases} 0, & \text{if } n \equiv\ 0\ (mod\ 4) \\ 2, & \text{if } n \equiv\ 1\ (mod\ 4) \\ 1, & \text{if } n \equiv\ 2\ (mod\ 4) \\ 3, & \text{if } n \equiv\ 3\ (mod\ 4) \end{cases}$$

For $n \equiv\ 0, 1, 3\ (mod\ 4)$ and $\xi \in \{1, 2, \ldots, n-3\}$

$$c'(v_\xi) = c(r^{'}_\xi) + c(r^{''}_\xi) \equiv \begin{cases} 0, & \text{if } \xi \equiv\ 1\ (mod\ 4) \\ 3, & \text{if } \xi \equiv\ 2\ (mod\ 4) \\ 2, & \text{if } \xi \equiv\ 3\ (mod\ 4) \\ 1, & \text{if } \xi \equiv\ 0\ (mod\ 4) \end{cases}$$

$$c'(v_{n-2}) = c(r^{'}_{n-3}) + c(r^{'}_{n-2}) + c(r^{''}_{n-2}) \equiv \begin{cases} 3, & \text{if } n \equiv\ 0, 3\ (mod\ 4) \\ 2, & \text{if } n \equiv\ 1\ (mod\ 4) \end{cases}$$

$$c'(v_{n-1}) = c(r^{'}_{n-2}) + c(r^{'}_{n-1}) + c(r^{''}_{n-1}) \equiv \begin{cases} 2, & \text{if } n \equiv\ 0\ (mod\ 4) \\ 1, & \text{if } n \equiv\ 1\ (mod\ 4) \\ 0, & \text{if } n \equiv\ 3\ (mod\ 4) \end{cases}$$

$$c'(v_n) = c(r^{'}_{n-1}) \equiv \begin{cases} 0, & \text{if } n \equiv\ 1\ (mod\ 4) \\ 1, & \text{if } n \equiv\ 2\ (mod\ 4) \\ 3, & \text{if } n \equiv\ 0, 3\ (mod\ 4) \end{cases}$$

For $n \equiv\ 2\ (mod\ 4)$ and $\xi \in \{2, \ldots, n-1\}$

$c(r_1) = 3,$

$$c(r_\xi) = \begin{cases} 2, & \text{if } \xi \equiv\ 2\ (mod\ 4) \\ 3, & \text{if } \xi \equiv\ 3\ (mod\ 4) \\ 0, & \text{if } \xi \equiv\ 0\ (mod\ 4) \\ 1, & \text{if } \xi \equiv\ 1\ (mod\ 4) \end{cases}$$

For $\xi \in \{1, 2, \ldots, n-1\}$

$$c(r^{'}_\xi) = \begin{cases} 3, & \text{if } \xi \equiv\ 1\ (mod\ 4) \\ 0, & \text{if } \xi \equiv\ 2\ (mod\ 4) \\ 1, & \text{if } \xi \equiv\ 3\ (mod\ 4) \text{ and } \xi = n-1 \\ 2, & \text{if } \xi \equiv\ 0\ (mod\ 4) \end{cases}$$

$$c(r^{''}_\xi) = \begin{cases} 0, & \text{if } \xi \equiv\ 1\ (mod\ 4) \\ 1, & \text{if } \xi \equiv\ 2\ (mod\ 4) \\ 2, & \text{if } \xi \equiv\ 3\ (mod\ 4) \\ 3, & \text{if } \xi \equiv\ 0\ (mod\ 4) \end{cases}$$

For $\xi \in \{3, 4, \ldots, n-1\}$

$c'(u_1) = 3, c'(u_2) = 1,$

$$c'(u_\xi) = c(r_{\xi-1}) + c(r_\xi) + c(r^{''}_{\xi-1}) \equiv \begin{cases} 2, & \text{if } \xi \equiv\ 3\ (mod\ 4) \\ 1, & \text{if } \xi \equiv\ 0\ (mod\ 4) \\ 0, & \text{if } \xi \equiv\ 1\ (mod\ 4) \\ 3, & \text{if } \xi \equiv\ 2\ (mod\ 4) \end{cases}$$

$c'(u_n) = c(r_{n-1}) + c(r^{''}_{n-1}) = 1 + 0 \equiv 1$

For $n \equiv\ 2\ (mod\ 4)$ and $\xi \in \{2, 3, \ldots, n-2\}$

$c'(v_1) = c(r^{'}_1) + c(r^{''}_1) = 0 + 3 \equiv 3$

$$c'(v_\xi) = c(r^{'}_\xi) + c(r^{''}_\xi) \equiv \begin{cases} 0, & \text{if } \xi \equiv\ 2\ (mod\ 4) \\ 3, & \text{if } \xi \equiv\ 3\ (mod\ 4) \\ 2, & \text{if } \xi \equiv\ 0\ (mod\ 4) \\ 1, & \text{if } \xi \equiv\ 1\ (mod\ 4) \end{cases}$$

$c'(v_{n-1}) = c(r^{'}_{n-2}) + c(r^{'}_{n-1}) + c(r^{''}_{n-1}) = 2 + 1 + 0 \equiv 3,$

$c'(v_n) = c(r^{'}_{n-1}) = 1.$

Hence c is a twin edge 4−coloring of SL_n and therefore $\chi_t{}'(SL_n) \leq 4$. By observation 1.2. This implies that $\chi_t{}'(SL_n) \geq 4$, and so $\chi_t{}'(SL_n) = 4$. □

Example 3.12. The graph SL_7 and its twin edge coloring is shown in Fig. 9.

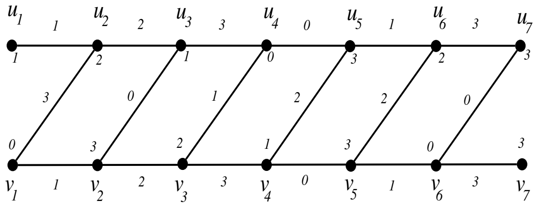

FIGURE 9. SL_7 and its twin edge coloring

Example 3.13. The graph SL_8 and its twin edge coloring is shown in Fig. 10.

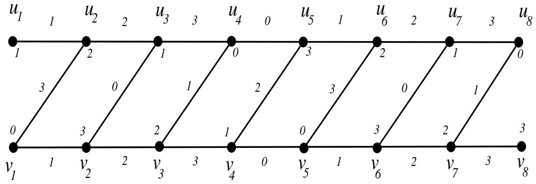

FIGURE 10. SL_8 and its twin edge coloring

Example 3.14. The graph SL_9 and its twin edge coloring is shown in Fig. 11.

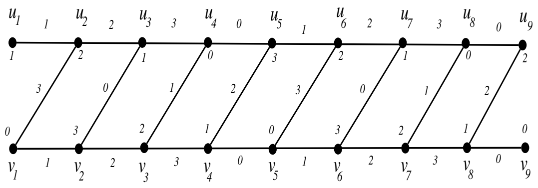

FIGURE 11. SL_9 and its twin edge coloring

Theorem 3.15. *Let $L(P_n^+)$, be the line graph of comb graph with $n \geq 4$, then* $\chi_t{}'(L(P_n^+) = 5$.

Proof. Consider $L(P_n^+)$. Let $V(L(P_n^+)) = \{u_\xi : 1 \leq \xi \leq n-1\} \cup \{v_\xi : 1 \leq \xi \leq n\}$ and $E(L(P_n^+)) = \{r_\xi : 1 \leq \xi \leq n-2\} \cup \{r_\xi^{'}, r_\xi^{''} : 1 \leq \xi \leq n-1\}$, where $r_\xi = u_\xi u_{\xi+1}, r_\xi^{'} = u_\xi v_\xi, r_\xi^{''} = u_\xi v_{\xi+1}$.
For $n = 4$, define $c : E(L(P_4^+)) \to Z_4$ as follows:
$c(r_1) = 1, c(r_2) = c(r_1^{'}) = 0, c(r_2^{'}) = c(r_3^{'}) = 3,$
$c(r_1^{''}) = c(r_2^{''}) = 2, c(r_3^{''}) = 1.$
The induced vertex coloring is:
$c'(u_1) = 3$; $c'(u_2) = 2$; $c'(u_3) = 0$; $c'(v_1) = 0$; $c'(v_2) = 1$; $c'(v_3) = 1$; $c'(v_4) = 1$; and it is proper. Hence $\chi_t^{'}(L(P_4^+)) \leq 4$. By observation 1.2, $\chi_t^{'}(L(P_4^+)) \geq 4$ and so $\chi_t^{'}(L(P_4^+)) = 4$.
For $n \geq 5$, define $c : E(L(P_n^+)) \to Z_5$ as follows:
For $\xi \in \{1, 2, \ldots, n-2\}$

$$c(r_\xi) \equiv \begin{cases} 2, & \text{if } \xi \equiv 1 \ (mod\ 2) \\ 4, & \text{if } \xi \equiv 0 \ (mod\ 2) \end{cases}$$

$$c(r_\xi^{'}) \equiv \begin{cases} 0, & \text{if } \xi \equiv 1 \ (mod\ 2) \\ 3, & \text{if } \xi \equiv 0 \ (mod\ 2) \end{cases}$$

$c(r_\xi^{''}) = 1; c(r_{n-1}^{''}) = 3;$
The induced vertex coloring is:
$c'(u_1) = c(r_1^{'}) + c(r_1^{''}) + c(r_1) = 0 + 1 + 2 \equiv 3$
$c'(v_1) = 0$
For $\xi \in \{2, 3, \ldots, n-2\}$

$$c'(u_\xi) = c(r_{\xi-1}) + c(r_\xi) + c(r_\xi^{'}) + c(r_\xi^{''}) \equiv \begin{cases} 0, & \text{if } \xi \equiv 0 \ (mod\ 2) \\ 2, & \text{if } \xi \equiv 1 \ (mod\ 2) \end{cases}$$

$c'(u_{n-1}) = c(r_{n-2}) + c(r_{n-1}^{'}) + c(r_{n-1}^{''}) = 2 + 0 + 3 \equiv 0$

$$c'(v_\xi) = c(r_{\xi-1}^{''}) + c(r_\xi^{'}) \equiv \begin{cases} 4, & \text{if } \xi \equiv 0 \ (mod\ 2) \\ 1, & \text{if } \xi \equiv 1 \ (mod\ 2) \end{cases}$$

$$c'(v_{n-1}) = 1; c'(v_n) = 3$$

Hence c is a twin edge 5−coloring of $L(P_n^+)$ and therefore ${\chi_t}'(L(P_n^+)) \leq 5$. By observation 1.2. This implies that ${\chi_t}'(L(P_n^+)) \geq 5$, and so ${\chi_t}'(L(P_n^+)) = 5$. □

Remark 3.1. If $n = 3$, then ${\chi_t}'(L(P_3^+)) = 4$.

Example 3.16. The graph $L(P_5^+)$ and its twin edge coloring is shown in Fig. 12.

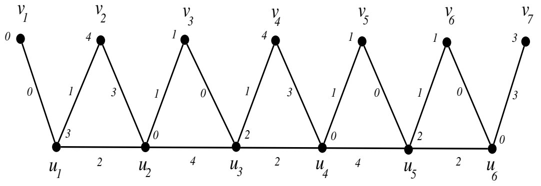

FIGURE 12. $L(P_5^+)$ and its twin edge coloring

Theorem 3.17. *Let* $L(T_n)$, *be the line graph of twig graph with* $n \geq 4$, *then* ${\chi_t}'(L(T_n)) = 7$.

Proof. Consider $L(T_n)$. Let $V(L(T_n)) = \{u_\xi : 1 \leq \xi \leq n\} \cup \{v_\xi, w_\xi : 1 \leq \xi \leq n-1\}$ and $E(L(T_n)) = \{r_\xi, r_\xi^{'}, s_\xi, s_\xi^{'}, t_\xi, t_\xi^{'} : 1 \leq \xi \leq n-2\}$, where $r_\xi = u_\xi u_{\xi+1}, r_\xi^{'} = v_\xi w_\xi, s_\xi = u_\xi v_\xi, s_\xi^{'} = u_\xi w_\xi, t_\xi = u_{\xi+1} v_\xi, t_\xi^{'} = u_{\xi+1} w_\xi$.
For $n = 4$, define $c : E(L(T_4)) \to Z_5$ as follows:
$c(r_1) = 5; c(r_2) = 0; c(r_1^{'}) = c(r_2^{'}) = 0; c(s_1) = c(s_2) = 1; c(s_1^{'}) = c(s_2^{'}) = 2; c(t_1) = c(t_2) = 3; c(t_1^{'}) = c(t_2^{'}) = 4$
The induced vertex coloring is:
$c'(u_1) = 2$; $c'(u_2) = 3$; $c'(u_3) = 1$; $c'(v_1) = 4$; $c'(v_2) = 4$; $c'(w_1) = 0$; $c'(w_2) = 0$ and it is proper. Hence ${\chi_t}'(L(T_4)) \leq 6$. By observation 1.2, ${\chi_t}'(L(T_4)) \geq 6$ and so ${\chi_t}'(L(P_4^+)) = 6$.
For $n \geq 5$, define $c : E(L(T_n)) \to Z_6$ as follows:
For $\xi \in \{1, 2, \ldots, n-1\}$

$$c(r_\xi) = \begin{cases} 5, & \text{if } \xi \equiv 1 \ (mod\ 3) \\ 6, & \text{if } \xi \equiv 2 \ (mod\ 3) \\ 0, & \text{if } \xi \equiv 0 \ (mod\ 3) \end{cases}$$

$c(r_\xi^{'}) = 0, c(s_\xi) = 1, c(t_\xi) = 3, c(s_\xi^{'}) = 2, c(t_\xi^{'}) = 4$
The induced vertex coloring is:
$c'(u_1) = c(r_1) + c(s_1) + c(t_1^{'}) = 5 + 1 + 2 \equiv 1$
For $\xi \in \{2, 3, \ldots, n-1\}$

$$c'(u_\xi) = c(r_{\xi-1})+c(r_\xi)+c(t_{\xi-1})+c(t'_{\xi-1})+c(s_\xi)+c(s'_\xi) \equiv \begin{cases} 1, & \text{if } \xi \equiv\ 1\ (mod\ 3) \\ 0, & \text{if } \xi \equiv\ 2\ (mod\ 3) \\ 2, & \text{if } \xi \equiv\ 0\ (mod\ 3) \end{cases}$$

$$c'(u_n) = c(r_{n-1}) + c(t_{n-1}) + c(t'_{n-1}) \equiv \begin{cases} 0, & \text{if } n \equiv\ 1\ (mod\ 3) \\ 5, & \text{if } n \equiv\ 2\ (mod\ 3) \\ 6, & \text{if } n \equiv\ 0\ (mod\ 3) \end{cases}$$

For $\xi \in \{1, 2, \ldots, n-1\}$
$c'(v_\xi) = 4, c'(w_\xi) = 6$
and it is proper. Hence ${\chi_t}'(L(T_n)) \leq 7$. By observation 1.2. This implies that ${\chi_t}'(L(T_n)) \geq 7$ and so ${\chi_t}'(L(T_n)) = 7$. □

Example 3.18. The graph $L(T_5)$ and its twin edge coloring is shown in Figure 13.

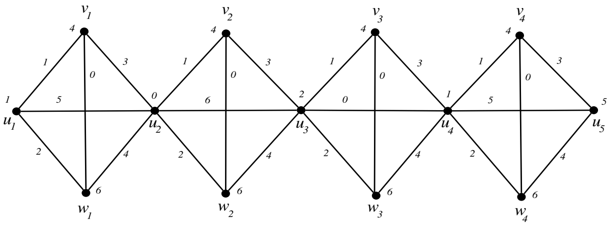

FIGURE 13. $L(T_5)$ and its twin edge coloring

Open Problem: The investigation of analogous results for different graphs and different operation of above families of graphs are still open

Conflicts of interest : The authors declare no conflicts of interest.

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REFERENCES

1. S. Anantharaman, *Twin edge coloring of total graph and graphs with twin chromatic index* $\Delta+2$, Applications and Applied Mathematics: An International Journal **15** (2019), 314-336.
2. E. Andrews, L. Helenius, D. Johnston, J. Verwys, P. Zhang, *On Twin edge coloring of graphs*, Discuss. Math. Graph Theory **34** (2014), 613-627.
3. E. Andrews, D. Johnston, P. Zhang, *A Twin edge coloring Conjecture*, Bulletin of the ICA **70** (2014), 28-44.
4. E. Andrews, D. Johnston, P. Zhang, *On twin edge colorings in trees*, J. Combin. Math. Combin. Comput. **94** (2015), 115-131.
5. G. Chartrand and P. Zhang, *Chromatic Graph Theory*, Chapman and Hall/CRC, Boca Raton, New York, 2008.

6. G. Jayaraman, D. Muthuramakrishnan and S. Vishnu Kumar, *Total coloring of line graph and square graph for certain graphs*, Advances and Application in Mathematical sciences **21** (2022), 6339-6344.
7. D. Johnston and P. Zhang, *An upper bound for the twin chromatic index of a graph*, Congr. Numer. **219** (2014), 175-182.
8. S. Lakshmi and V. Kowsalya, *Twin edge colourings of wheel graphs*, IOSR J. Math. **12** (2016), 71-73.
9. J. Naveen, S. Meena, *Twin edge coloring of some path and cycle related graphs*, Asian Journal of Mathematics and Computer Research **28** (2021), 38-57.
10. J. Naveen, *Twin edge coloring of product graphs*, Asian Research Journal of Current Science **4** (2022), 62-72.
11. R. Rajarajachozhan, R. Sampathkumar, *Twin edge colorings of certain square graphs and product graphs*, Electronic Journal of Graph Theory and Applications **4** (2016), 79-93.
12. J. Tolentino, R. Marcelo, and M.A. Tolentino, *Twin chromatic indices of some graphs with maximum degree 3*, J. Phys.: Conf. Ser. **1538** (2020).
13. J. Tolentino, R Marcelo, Mark Anthony Tolentino, *On twin edge colorings in m-ary trees*, Electronic Journal of Graph Theory and Applications **10** (2022), 131-149.
14. Q. Yang, S.L. Tian, L.W.Q. Suo, *Twin edge coloring of infinite lattices*, IOP Conf. Series: Journal of Physics: Conf. Series **1087** (2018), 052020.

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