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The Forcing Circular Metric Dimension of A Graph

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Abstract

Let $G = (V, E)$ be a simple graph and u, v be any two vertices of G. Then the circular distance between u and v denoted by $D^c(u, v)$ and is defined by

$$D^c(u,v) = \begin{cases} D(u,v) + d(u,v) & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

where $D(u,v)$ and $d(u,v)$ are detour distance and distance between u and v respectively. Let $W = \{w_1, w_2, \ldots, w_k\} \subset V(G)$ and $v \in V(G)$. The representation $cr(v/W)$ of v with respect to W is the k-tuple $\left(D^c(v, w_1), D^c(v, w_2), \ldots, D^c(v, w_k)\right)$.Then W is called a circular resolving set if different vertices of G have different representations with respect to W. A circular resolving set of minimum cardinality is called a $cdim$ -set for G and the cardinality of the $cdim$ -set is known as the *circular metric dimension* of G, represented by $cdim(G)$. Let W be a minimum $cdim$ -set of G. A subset of $T \subseteq W$ is called a forcing subset of W if W is the unique minimum $cdim$ -set containing T. A forcing circular resolving subset for W of minimum cardinality is a minimum forcing circular resolving subset of W. The forcing circular metric dimension of W, $f_{cdim}(W)$ in G is the cardinality of a minimum forcing circular resolving subset of W. The forcing circular metric dimension of , $f_{cdim}(G) = \min\{f_{cdim}(W)\}$,where the minimum is taken over all minimum circular resolving subset of W. In this article forcing circular metric dimension of some standard graphs are determined.

Keywords: circular distance, circular resolving set, circular metric dimension, forcing circular resolving set, forcing circular metric dimension.

AMS Subject Classification: 05C12

1. Introduction and Preliminaries

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The order of a graph G is $|V(G)|$, its number of vertices denoted by n. The size of a graph G is $|E(G)|$, its number of edges denoted by m. For basic graph theory terminology, we refer [2,5]. The *degree* $\deg(v)$ of a vertex $v \in V(G)$ is the number of edges incident to v. We denote by $\Delta(G)$ the maximum degree of a graph G. The *distance* $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them. The detour *distance* $D(u,v)$ between two vertices $u,v \in V(G)$ is the length of a longest path between them. These concepts were studied in [1,6]. Let $G = (V,E)$ be a simple graph and u,v be any two vertices of G. Then the circular distance between u and v denoted by $D^c(u,v)$ and is defined by

$$D^c(u,v) = \begin{cases} D(u,v) + d(u,v) & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

where $D(u,v)$ and $d(u,v)$ are detour distance and distance between u and v respectively. Let $W = \{w_1, w_2, \ldots, w_k\} \subset V(G)$ and $v \in V(G)$. The representation $cr(v/W)$ of v with respect to W is the k-tuple $\left(D^c(v,w_1), D^c(v,w_2), \ldots, D^c(v,w_k)\right)$.Then W is called a circular resolving set if different vertices of G have different representations with respect to W. A circular resolving set of minimum cardinality is called a $cdim$ -set for G and the cardinality of the $cdim$ -set is known as the *circular metric dimension* of G, represented by $cdim(G)$. These concepts were studied in [3,4,7,8]. In this article, we study a new circular metric dimension called the forcing circular metric dimension of a graph.

2. The forcing circular metric dimension of a graph

Definition 2.1. Let W be a minimum $cdim$ -set of G. A subset of $T \subseteq W$ is called a forcing subset of W if W is the unique minimum $cdim$ -set containing T. A forcing circular resolving subset for W of minimum cardinality is a minimum forcing circular resolving subset of W. The forcing circular metric dimension of W, $f_{cdim}(W)$ in G is the cardinality of a minimum forcing circular resolving subset of W. The forcing circular metric dimension of , $f_{cdim}(G) = \min\{f_{cdim}(W)\}$,where the minimum is taken over all minimum circular resolving subset of W.

Example 2.2.

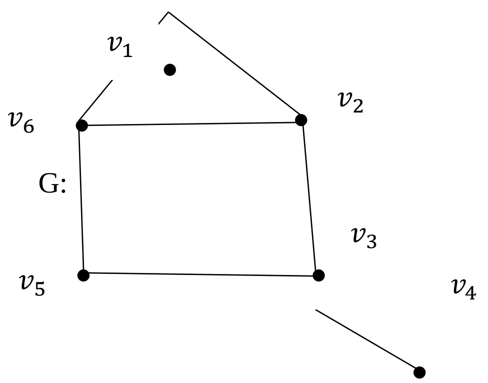

Figure 2.1

For the graph G given in Figure 2.1, let W=$\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then

cr$(v_1/W) = (0,5,5,5,5,5)$

cr$(v_2/W) = (5,0,5,7,5,4)$

cr$(v_3/W) = (5,5,0,2,5,5)$

cr$(v_4/W) = (7,7,2,0,7,7)$

cr$(v_5/W) = (5,5,5,7,0,5)$

cr$(v_6/W) = (5,4,5,7,5,0)$

Clearly W_1=$\{v_1, v_2, v_3\}$, W_2= $\{v_1, v_2, v_4\}$, W_3= $\{v_1, v_2, v_5\}$, W_4= $\{v_2, v_3, v_4\}$, W_5= $\{v_2, v_3, v_5,\}$, W_6= $\{v_1, v_5, v_6\}$, W_7= $\{v_4, v_5, v_6\}$, W_8= $\{v_1, v_3, v_6\}$, W_9= $\{v_1, v_4, v_6\}$, W_{10}= $\{v_3, v_4, v_6\}$, W_{11}= $\{v_3, v_5, v_6\}$,W_{12}= $\{v_2, v_4, v_5\}$ are the only cdim-set of G such that $f_{cdim}(W_i)$=3, so that $f_{cdim}(G) = 3$

Observation: 2.3. For a connected graph G, $0 \leq f_{cdim}(G) \leq cdim(G)$.

Theorem 2.4. For the path $G = P_n (n \geq 2)$, $f_{cdim}(G) = 1$.

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Then $W_1 = \{v_1\}$ and $W_2 = \{v_n\}$ are the only two $cdim$-sets of G such that $f_{cdim}(W_1) = f_{cdim}(W_2) = 1$. Hence $f_{cdim}(G) = 1$. ■

Theorem 2.5. For the cycle $G = C_n (n \geq 3)$, $f_{cdim}(G) = n - 1$.

Proof. Let $(C_n) = \{v_1, v_2, \dots, v_n\}$. It is easily seen that $W = V(G) - \{x\}$, where $x \in V(G)$ is a $cdim$-set of G so that $f_{cdim}(W) = n - 1$. Since x is arbitrary, $f_{cdim}(G) = n - 1$. ■

Theorem 2.6. For the complete graph $G = K_n, n \geq 2$, $f_{cdim}(G) = n - 1$.

Proof. The proof is similar to the Theorem 2.5. ■

Theorem 2.7. Let G be the graph obtained from $K_{1,n-1} (n \geq 3)$ by subdividing the end edges exactly once. Then $f_{cdim}(G) = n - 2$.

Proof. It is easily seen that $W = V(G) - \{x, u\}$, where $u \in \{v_1, v_2, \dots, v_{n-1}\}$ is a $cdim$-set of G and so $f_{cdim}(W) = n - 2$. Since u is arbitrary, $f_{cdim}(G) = n - 2$. ■

Theorem 2.8. For the graph $G = K_{n_1} + P_{n_2}$, where $n_1, n_2 \geq 2$. $f_{cdim}(G) = 1$.

Proof. Let $V(K_{n_1}) = \{u_1, u_2, \dots, u_{n_1}\}$ and $V(P_{n_2}) = v_1, v_2, \dots, v_{n_2}$. Let $S = V(K_{n_1}) \cup \{x\}$, where $x \in V(P_{n_2})$. Without loss of generality, let $x = u_1$. $cr(u_2/S) = cr(u_{n_1}/S) =$ Then S is a $cdim$-set of G so that $cdim(G) = n_1 + 1$. By Theorem $f_{cdim}(G) \leq \dim(G) - |V(K_{n_1})| = n + 1 - n = 1$. Since $cdim$-set of G is not unique , $f_{cdim}(G) = 1$. ■

Theorem 2.9. For the complete bipartite graph $G = K_{r,s}$, $(1 \leq r \leq s)$,

$$f_{cdim}(G) = \begin{cases} 1; r = 1, 1 \leq s \leq 2, \\ r + s - 2;\ r = 1, s \geq 3. \\ r + s - 1; 2 \leq r \leq s \end{cases}$$

Proof. Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be the two bipartite sets of G. We have the three cases.

Case (i): $r = 1, 1 \leq s \leq 2$. The result follows from Theorem 2.4.

Case (ii): $r = 1, s \geq 3$.

Let $W = V(G) - \{x_1, y_s\}$. Then the circular metric representations $(n-2)$ tuples are as follows:

$$cr(x_1/W) = (2,2,2, \dots ,2,2)$$
$$cr(y_1/W) = (0,4,4, \dots ,4,4)$$
$$cr(y_2/W) = (4,0,4, \dots ,4,4)$$
$$\vdots$$
$$cr(y_{s-1}/W) = (4,4,4, \dots ,4,0)$$
$$cr(y_s/W) = (4,4,4, \dots ,4,4).$$

Since the representation are distinct, W is a circular resolving set of G so that $cdim(G) \leq r + s - 2$. We demonstrate that $cdim(G) = r + s - 2$. On the other hand, imagine that, $cdim(G) \leq r + s - 3$. Then there exists a circular resolving set S' such that. $|S'| \leq r + s - 3$. As a result, there are at least two end vertices $u, v \in V \backslash S'$ such that $cr(u/S') = cr(v/S') = (4,4,4, \dots ,4,4)$, which is incoherent. As a result, $cdim(G) = r + s - 2$. It is easily seen that $W = V(G) - \{x, u\}$, where $u \in \{y_1, y_2, \dots, y_{s-1}\}$ is a $cdim$-set of G and so $f_{cdim}(W) = r + s - 2$. Since u is arbitrary, $f_{cdim}(G) = r + s - 2$.

Cases (iii): $2 \leq r \leq s$.

Let $W = V(G) - \{y_s\}$. Then the circular metric representations $(r + s - 1)$ tuples are as follows:

$$cr(x_1/W) = (0, r+s-1, r+s-1, \dots, r+s-1)$$
$$cr(x_2/W) = (r+s-1, 0, r+s-1, \dots, r+s-1)$$
$$\vdots$$
$$cr(x_r/W) = (r+s-1, r+s-1, \dots, 0, r+s-1, \dots, r+s-1)$$

→ r^{th} place

$$cr(y_1/W) = (r+s-1, r+s-1, \dots, r+s-1, 0, r+s-1 \dots, r+s-1)$$

→ $(r+1)^{th}$ place

$$cr(y_2/W) = (r+s-1, r+s-1, \dots, r+s-1, 0,\ r+s-1\ \dots, r+s-1)$$

→ $(r+2)^{th}$ place

$$\vdots$$

.

$$cr(y_{s-1}/W) = (r+s-1, r+s-1, r+s-1, \dots, r+s-1, r+s-1, \dots, 0)$$

↓

$(r+s-1)^{th}$ place

$$cr(y_s/W) = (r+s-1, r+s-1, r+s-1, \dots, r+s-1, r+s-1, \dots, r+s-1).$$

Since the representation are distinct, W is a circular resolving set of G so that $cdim(G) \le r+s-1$. We demonstrate that $cdim(G) = r+s-1$. Consider however, that $cdim(G) \le r+s-2$. If so, a circular resolving set S' exists such that $|S'| \le r+s-2$ As a result, there are at least two vertices, $u, v \in V \backslash S'$such that $cr(u/S') = cr(v/S') = (r+s-1, r+s-1, r+s-1, \dots, r+s-1)$, which is incoherent. As a result, $cdim(G) = r+s-1$. It is easily seen that $W = V(G) - \{y_s\}$, where $y_s \in V(G)$ is a $cdim$-set of G so that $f_{cdim}(W) = r+s-1$. Since y_s is arbitrary, $f_{cdim}(G) = r+s-1$. ■

Theorem 2.10. For the graph $G = C_n \circ K_1$, $(n \ge 4)$, $f_{cdim}(G) = n$.

Proof. Let $V(C_n) = \{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$ be the set of end vertices of G. Let $W = \{v_1, v_2, \dots, v_n\}$. Then the circular metric representations n tuples are as follows

$$cr(v_1/W) = (0, n+4, n+4, \dots, n+4)$$
$$cr(v_2/W) = (n+4, 0, n+4, \dots, n+4)$$

.
..

$$cr(v_n/W) = (n+4, n+4, \dots n+4, 0)$$
$$cr(u_1/W) = (2, n+2, n+2, n+2, \dots, n+2)$$
$$cr(u_2/W) = (n+2, 2, n+2, \dots, n+2)$$
$$cr(u_3/W) = (n+2, n+2, 2, \dots, n+2)$$

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$$cr(u_n/W) = (n+2, n+2, \dots, n+2, 2).$$

W is a circular resolving set of G since the representations are distinct and as a result, $cdim(G) \le n$. We establish that $cdim(G) = n$. Consider, however, that $cdim(G) \le n-1$. If so, a circular resolving set S' exists such that $|S'| \le n-1$. Therefore, either two end vertices or at least two cut vertices of G belongs to $V \backslash S'$. Allow $u, v \in V/S'$. If G's end vertices are u and v, then $cr(u/S') = cr(v/S') = (n+2, n+4, n+4, \dots, n+4)$. If G's cut vertices are u and v, then $cr(u/S') = cr(v/S') = (n, n+2, n+2, \dots, n+2)$. Which is incongruous. Consequently, $cdim(G) = n$. It is easily seen that $W = \{v_1, v_2, \dots, v_n\}$. , where $v_i \in V(G)$ is a $cdim$-set of G so that $f_{cdim}(W) = n$. Since v_i is arbitrary, $f_{cdim}(G) = n$.

■

Theorem 2.11. For the graph $G = K_n \circ K_1$, $(n \ge 4)$, $f_{cdim}(G) = n$.

Proof. The proof is similar to the Theorem 2.10. ∎

Theorem 2.12. Let G be the middle graph of the path $P_n (n \geq 3)$. Then $f_{cdim}(G) = 0$.

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $u_i = v_i v_{i+1}, 1 \leq i \leq n-1$. Then $(G) = \{v_i,\ u_j,\}(1 \leq i \leq n, 1 \leq j \leq n-1)$. In G, v_i is adjacent to u_1, v_n is adjacent to u_{n-1}. v_i is adjacent to u_{i-1}, u_i, $2 \leq i \leq n-1$, u_1 is adjacent to v_1, v_2 and u_2, u_i is adjacent to u_{i-1}, u_{i+1}, v_i and v_{i+1} for$2 \leq i \leq n-2$ and u_{n-1} is adjacent to $v_{n-2} v_{n-1}$ and v_n.

Evidently, $|V(G)| = 2n-1$. As a result of $G \neq P_n$, by Theorem 2.8, $cdim(G) \geq 2$. Let $W = \{v_1, v_n\}$, $Cr(v_1/W) = (0, 3n-2)$, $Cr(v_2/W) = (5{,}3n-4), \ldots, Cr(v_{n-1}/W) = (3n-5{,}5), Cr(v_n/W) = (3n-2, 0)$, $Cr(u_1/W) = (2, 3n-4), Cr(u_2/W) = (5{,}3n-7)$, $Cr(u_3/W) = (8{,}3n-10), \ldots, Cr(u_{n-1}/W) = (3n-4, 2)$. W is a circular resolving set of G, and since the circular metric representation is distinct, $cdim(G) = 2$. It is easily seen that $W = \{v_1, v_n\}$ is an unique $cdim$-set of G so that $f_{cdim}(W) = 0\ and\ hence\ f_{cdim}(G) = 0$

∎

Theorem 2.13. Let G be the total graph of the path $P_n (n \geq 4)$. Then $f_{cdim}(G) = 0$.

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $u_i = v_i v_{i+1}, 1 \leq i \leq n-1$. Then $(G) = \{v_i,\ u_j, 1 \leq i \leq n, 1 \leq j \leq n-1\}$. In G, v_1 is adjacent to v_2 and u_1; v_n is adjacent to v_{n-1} and u_{n-1}; v_i is adjacent to $v_{i-1}, v_{i+1}, u_{i-1}$ and u_i; $2 \leq i \leq n-1$, u_1 is adjacent to v_1,v_2 and u_2, u_i is adjacent to u_{i-1}, u_{i+1}, v_i and v_{i+1} for $2 \leq i \leq n-2$ and u_{n-1} is adjacent to v_{n-2}, v_n and u_{n-2}. Evidently, $|V(G)| = 2n-1$. As a result of $G \neq P_n$, $cdim(G) \geq 2$. Let $W = \{v_1, v_n\}$, $Cr(v_1/W) = (0{,}3n-3), Cr(v_2/W) = (2n-1{,}3n-4), Cr(v_3/W) = (2n, 3n-5)$, $\ldots, Cr(v_{n-1}/W) = (3n-4{,}2n-1), Cr(v_n/W) = (3n-3{,}0), Cr(u_1/W) = (2n-1,\ 3n-3), Cr(u_2/W) = (2n, 3n-4), Cr(u_3/W) = (2n+1{,}3n-5), \ldots$,

$Cr(u_{n-1}/W) = (3n-4{,}2n), Cr(u_n/W) = (3n-3{,}2n-1). W$ is a circular resolving set of G, and since the circular metric representation is distinct, $cdim(G) = 2$. It is easily seen that $W = \{v_1, v_n\}$ is an unique $cdim$-set of G so that $f_{cdim}(W) = 0\ and\ hence\ f_{cdim}(G)$ =0. ∎

Conclusion

This article established a novel distance metric called the forcing circular metric dimension of a graph. We will develop this concept to incorporate more distance considerations in a subsequent investigation.

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