The connected circular metric dimension of a graph

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Abstract

Let $G = (V, E)$ be a simple graph and u, v be any two vertices of G. Then the circular distance between u and v denoted by $D^c(u, v)$ and is defined by

$$D^c(u,v) = \begin{cases} D(u,v) + d(u,v) & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

where $D(u, v)$ and $d(u, v)$ are detour distance and distance between u and v respectively. Let $W = \{w_1, w_2, \dots, w_k\} \subset V(G)$ and $v \in V(G)$. The representation $cr(v/W)$ of v with respect to W is the k-tuple $\left(D^c(v, w_1), D^c(v, w_2), \dots, D^c(v, w_k)\right)$.Then W is called a circular resolving set if different vertices of G have different representations with respect to W. A circular resolving set W is called connected circular resolving set, $G[W]$ is connected . The minimum cardinality of a connected circular resolving set in a graph G is its connected circular metric dimension of G and is denoted by $cdim_c(G)$. The connected circular metric dimension of some standard graphs are determined. Some general properties satisfies by this concept are studied. Connected graphs of order $n \geq 3$ with connected circular metric dimension 1 are characterized. Necessary condition for the connected circular metric dimension to be $n - 1$ is given.

Keywords: distance, detour distance, circular distance, circular resolving set, connected circular metric dimension.

AMS Subject Classification: 05C12.

1. Introduction and Preliminaries

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The order of a graph G is $|V(G)|$, its number of vertices denoted by n. The size of a graph G is $|E(G)|$, its number of edges denoted by m. For basic graph theory terminology, we refer [3]. The *degree*, $\deg(v)$ of a vertex $v \in V(G)$ is the number of edges incident to v. We denote $\Delta(G)$ the maximum degree of a graph G. The *distance* $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them. These concepts were studied in [1,2,5,10-14,17-24,26,27]. The detour *distance* $D(u,v)$ between two vertices $u,v \in V(G)$ is the length of a longest path between them. These concepts were studied in [6-8,15]. Let $W = \{w_1, w_2, \ldots, w_k\} \subset V(G)$ and $v \in V(G)$. The representation $r(v/W)$ of v with respect to W is the k-tuple $\big(d(v,w_1), d(v, w_2), \ldots, d(v,w_k)\big)$.Then W is called a *resolving set* if different vertices of G have different representations with respect to W. A resolving set of minimum number of elements is called a *basis* for G and the cardinality of the basis is known as the *metric dimension* of G, represented by $dim(G)$. These concepts were studied in [4,28]. The representation $Dr(v/W)$ of v with respect to W is the k-tuple $\big(D(v,w_1), D(v, w_2), \ldots, D(v,w_k)\big)$.Then W is called a *detour resolving set* if different vertices of G have different representations with respect to W. A detour resolving set of minimum number of elements is called a *detour basis* for G and the cardinality of the basis is known as the *detour metric dimension* of G, represented by $Ddim(G)$.

The circular distance between u and v is denoted by $D^c(u,v)$ and is defined by

$$D^c(u,v) = \begin{cases} D(u,v) + d(u,v) & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

An $u-v$ path of length $D^c(u,v)$ is called a $u-v$ circular. The circular diameter is the maximum circular distance be a pair of vertices of G. It is denoted by the $D^c(G)$. A circular path of length $D^c(G)$ is called the circular diametral path. These concepts were studied in [16].

Let $W = \{w_1, w_2, \ldots, w_k\} \subset V(G)$ and $v \in V(G)$. The representation $cr(v/W)$ of v with respect to W is the k-tuple $\big(D^c(v,w_1), D^c(v, w_2), \ldots, D^c(v,w_k)\big)$.Then W is called a *circular resolving set* if different vertices of G have different representations with respect to W. A circular resolving set of minimum number of elements is called a circular basis for G and the cardinality of the basis is known as the *circular metric dimension* of G, represented by $cdim(G)$. These concepts were studied in [25]. In this article, we study a new metric dimension called the connected circular metric dimension of a graph.The following theorem is used in the sequel.

Theorem 1.3. [28]] For the complete graph $G = K_n$ $(n \geq 2)$, $cdim(G) = n - 1$.

2. The connected circular metric dimension of a graph

Definition 2.1. A circular resolving set W is called a *connected circular resolving set* of G if $G[W]$ is connected. The minimum cardinality of a connected circular resolving set in a graph G is its *connected circular metric dimension* of G and is denoted by $cdim_c(G)$.

Example 2.2. For the graph G given in Figure 2.1, let W= $\{v_1, v_2\}$. Then $cr(v_1/W) = (0, 4)$, $cr(v_2/W)$ = (4,0), $cr(v_3/W)$ = (3,4), $cr(v_4/W)$ = (4,5). Since $cr(v/W)$ are distinct for all $v \in V(C_n)$, it follows that W is a circular resolving set of G. Since $G[W]$ is connected, W is a connected circular resolving set of G and so $cdim_c(G) \leq 2$. Also since no singleton subset of $V(G)$ is a circular resolving set of G, we have $cdim_c(G) = 2$.

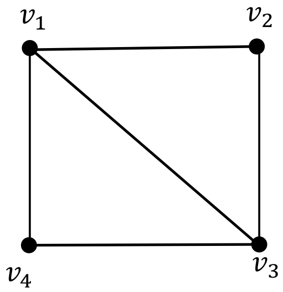

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Figure 2.1

Example 2.3. For the graph G given in Figure 2.2, no singleton subset of $V(G)$ is a circular resolving set of G, we have $cdim_c(G) \geq 2$. Let $W = \{v_1, v_3\}$. Then $cr(v_1/W)$ = (0,4), $cr(v_2/W)$ = (2,2), $cr(v_3/W)$ = (4,0), $cr(v_4/W)$ = (4,4). Since $cr(v/W)$ are distinct for all $v \in V(C_n)$, it follows that W is a circular resolving set of G. Since $G[W]$ is not connected, W is not a connected circular resolving set of G. It is easily verified that no two-element subset of $V(G)$ is not a circular resolving set of G and so $cdim_c(G) \geq 3$. Let $W_1 = \{v_1, v_2, v_3\}$. Then W_1 is a connected circular resolving set of G so that $cdim_c(G) = 3$.

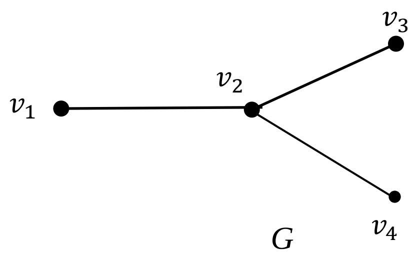

Figure 2.2

Observation 2.4. (i) Let G be a connected graph of order $n \geq 2$. Then $1 \leq cdim_c(G) \leq n-1$.
(ii) Each cut vertex of G belongs to every connected resolving set of G.

In the following we determine the connected circular metric dimension of some standard graphs.

Theorem 2.5. For the graph $G = P_n$ $(n \geq 2)$, $cdim_c(G) = 1$.

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and let $W = \{v_1\}$. Then $D^c(v_1, v_i) = 2(i-1), (1 \leq i \leq n)$. Since $cr(v/W)$ is distinct for all $v \in V(P_n)$, it follows that W is a circular resolving set of G. Also, $G[W]$ is connected, Hence W is a connected circular resolving set of G so that $cdim_c(G) = 1$.

Theorem 2.6. For the cycle $G = C_n, n \geq 3$, $cdim_c(G) = n-1$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and let $W = \{v_1, v_2, \dots, v_{n-1}\}$, The circular representations of $(n-1)$ tuples are as follows

$$cr(v_1/W) = (0, n, n, \dots, n),$$
$$cr(v_2/W) = (n, 0, n, n, \dots, n)$$
$$cr(v_3/W) = (n, n, 0, n, n, \dots, n)$$
$$\vdots$$
$$cr(v_{n-1}/W) = (n, n, n, \dots, n, 0)$$
$$cr(v_n/W) = (n, n, n, n, \dots, n).$$

Since $cr(v/W)$ are distinct for all $v \in V(C_n)$, it follows that W is a circular resolving set of G. Since $G[W]$ is connected, W is a connected circular resolving set of G. Therefore $cdim_c(G) \leq n-1$. We substantiate that $cdim_c(G) = n-1$.Consider, however, that $cdim_c(G) \leq n-2$.

Then, a set S'exists such that $|S'| \leq n-2$. As a result, there are at least two vertices u, v that satisfy the contradiction $cr(u/S') = cr(v/S') = (n, n, \dots, n)$.Consequently, $cdim_c(G) = n-1$.

Theorem 2.7. For the complete graph $G = K_n, n \geq 2,\ cdim_c(G) = n-1$.

Proof. The proof is similar to the Theorem 2.6.

Theorem 2.8. For the wheel graph $G = W_n, n \geq 3,\ cdim_c(G) = n-4$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_{n-1}, u\}$ and $W = \{v_1, v_2, \dots, v_{n-4}\}$, Then the circular representations of $(n-4)$ tuples are as follows

$$cr(v_1/W) = (0, n, n+1, \dots, n+1, n+1)$$
$$cr(v_2/W) = (n, 0, n, n+1, \dots, n+1, n+1)$$
$$cr(v_3/W) = (n+1, n, 0, n, n+1, \dots, n+1)$$

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$$cr(v_{n-4}/W) = (n+1, n+1, n+1, \dots, n, 0)$$
$$cr(v_{n-3}/W) = (n+1, n+1, n+1, \dots, n+1, n)$$
$$cr(v_{n-2}/W) = (n+1, n+1, n+1, \dots, n+1, n+1)$$
$$cr(v_{n-1}/W) = (n, n+1, n+1, \dots, n+1, n+1)$$
$$cr(u/W) = (n, n, n, n, \dots, n, n).$$

Since $cr(v/W)$ are distinct for all $v \in V(W_n)$, it follows that W is a circular resolving set of G. Since $G[W]$ is connected, W is a connected circular resolving set of G. Therefore $cdim_c(G) \leq n-4$. We substantiate that $cdim_c(G) = n-4$. Consider, however, that $cdim_c(G) \leq n-5$. Then, a set W'exists such that $|W'| \leq n-5$. As a result, there are at least two vertices u, v that satisfy the contradiction

$$cr(u/S') = cr(v/S') = (n+1, n+1, \dots, n+1).$$

Consequently, $cdim_c(G) = n-4$.

Theorem 2.9. For the complete bipartite graph $G = K_{r,s}$, $(1 \leq r \leq s)$,

$$cdim_c(G) = \begin{cases} 1; r = 1, 1 \leq s \leq 2, \\ r+s-2;\ r = 1, s \geq 3. \\ r+s-1; 2 \leq r \leq s \end{cases}$$

Proof. Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be the two bipartite sets of G. We have the three cases.

Case (i):$r = 1, 1 \leq s \leq 2$. The result follows from Theorem 2.5.

Case (ii): $r = 1, s \geq 3$.

Let $W = V(G) - \{x_1, y_s\}$. Then the circular metric representations $(n-2)$ tuples are as follows:

$$cr(x_1/W) = (2,2,2, \dots ,2,2)$$
$$cr(y_1/W) = (0,4,4, \dots ,4,4)$$
$$cr(y_2/W) = (4,0,4, \dots ,4,4)$$

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$$cr(y_{s-1}/W) = (4,4,4, \dots ,4,0)$$
$$cr(y_s/W) = (4,4,4, \dots ,4,4).$$

Since the representation are distinct and *G[W]* is connected, W is a connected circular resolving set of G so that $cdim_c(G) \leq r + s - 2$. We demonstrate that $cdim_c(G) = r + s - 2$. On the other hand, imagine that $cdim_c(G) \leq r + s - 3$. Then there exists a circular resolving set W' such that. $|W'| \leq r + s - 3$. As a result, there are at least two end vertices $u, v \in V \backslash W'$ such that $cr(u/W') = cr(v/W') = (4,4,4, \dots ,4,4)$, which is incoherent. As a result, $cdim_c(G) = r + s - 2$.

Cases (iii): $2 \leq r \leq s$.

Let $W = V(G) - \{y_s\}$. Then the circular metric representations $(r+s-1)$ tuples are as follows:

$$cr(x_1/W) = (0, r+s-1, r+s-1, \dots , r+s-1)$$
$$cr(x_2/W) = (r+s-1, 0, r+s-1, \dots , r+s-1)$$

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$$cr(x_r/W) = (r+s-1, r+s-1, \dots , 0, r+s-1, \dots , r+s-1)$$

r^{th} place

$$cr(y_1/W) = (r+s-1, r+s-1, \dots , r+s-1, 0, r+s-1 \dots , r+s-1)$$

$(r+1)^{th}$ place

$$cr(y_2/W) = (r+s-1, r+s-1, \dots , r+s-1, 0, \ r+s-1 \ \dots , r+s-1)$$

$(r+2)^{th}$ place

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$$cr(y_{s-1}/W) = (r+s-1, r+s-1, r+s-1, \dots , r+s-1, r+s-1, \dots ,0)$$

$(r+s-1)^{\text{th}}$ place

$$cr(y_s/W) = (r+s-1, r+s-1, r+s-1, \dots, r+s-1, r+s-1, \dots, r+s-1).$$

Since the representation are distinct and *G[W]* is connected, W is a connected circular resolving set of G so that $cdim_c(G) \leq r+s-1$. We demonstrate that $cdim_c(G) = r+s-1$. Consider however, that $cdim_c(G) \leq r+s-2$. If so, a circular resolving set W' exists such that $|W'| \leq r+s-2$ As a result, there are at least two vertices, $u, v \in V \backslash W'$such that $cr(u/W') = cr(v/W') = (r+s-1, r+s-1, r+s-1, \dots, r+s-1)$, which is incoherent. As a result, $cdim_c(G) = r+s-1$.

Theorem 2.10. Let G be the graph obtained from $K_{1,n-1}$, $(n \geq 3)$, by subdividing the end edges exactly once. Then $cdim_c(G) = n-1$.

Proof. Let x be the central vertex of $K_{1,n-1}$ $(n \geq 4)$ and $\{v_1, v_2, \dots, v_{n-1}\}$ be the set of end vertices of G. G is the graph obtained form $K_{1,n-1}$, $(n \geq 4)$, by subdividing xv_i $(1 \leq i \leq n-1)$ by u_i $(1 \leq i \leq n-1)$. Let $W = \{x, u_1, u_2, \dots, u_{n-2}\}$. Then

$$cr(x/W) = (0,2,2, \dots, 2,2)$$
$$cr(u_1/W) = (2,0,4,4, \dots, 4,4)$$
$$cr(u_2/W) = (2,4,0,4, \dots, 4,4)$$

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$$cr(u_{n-3}/W) = (2,4,4,4, \dots, 0,4,4)$$
$$cr(u_{n-2}/W) = (2,4,4,4, \dots, 4,4,0)$$
$$cr(u_{n-1}/W) = (2,4,4, \dots, 4,4,4)$$
$$cr(v_1/W) = (4,2,6,6,6, \dots, 6,6)$$
$$cr(v_2/W) = (4,6,2,6,6, \dots, 6,6)$$

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$$cr(v_{n-3}/W) = (4,6,6, \dots, 2,6)$$
$$cr(v_{n-2}/W) = (4,6,6,6, \dots, 6,2)$$
$$cr(v_{n-1}/W) = (4,6,6,6, \dots, 6,6,).$$

Due to the distinctness of the representations, W is a circular resolving set of G. Also $G[W]$ is connected, W is a connected circular resolving set of G so that $cdim_c(G) \leq n-1$. We substantiate that $cdim_c(G) = n - 1$. Consider, however, that $cdim_c(G) \leq n - 2$. If so, a circular resolving set W' exists such that $|W'| \leq n - 2$ and $G[W']$ is disconnected. Consequently, $cdim_c(G) = n - 1$.

Theorem 2.11. Let G be the graph obtained from C_n, $(n \geq 3)$, by subdividing the edges exactly once. Then $cdim_c(G) = 2n - 1$.

Proof: Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $\{u_1, u_2, \dots, u_n\}$ be the subdivided vertices of C_n. Then G is a cycle contains $2n$ vertices. By Theorem 2.6, $cdim_c(G) = 2n - 1$.

Theorem2.12. For the crown graph $G = H_{n,n}$, $n \geq 3$, $cdim_c(G) = n$.

Proof. Let $V(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and $E(G) = \left\{(u_i, v_j); 1 \leq i, j \leq n; i \neq j\right\}$,

Let $W = \{u_1, u_3, u_4, \dots, u_n, , v_2\}$. Then the circular representations of n tuples are as follows

$$cr(u_1/W) = (0, n, n, n, \dots, n, n, n, n)$$
$$cr(u_2/W) = (n, n, n, n, \dots, n, n, n, n + 2)$$
$$cr(u_3/W) = (n, 0, n, n, \dots, n, n, n, n)$$
$$cr(u_4/W) = (n, n, 0, n, \dots, n, n, n, n)$$

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$$cr(u_{n-1}/W) = (n, n, n, n, \dots, n, 0, n, n)$$
$$cr(u_n/W) = (n, n, n, n, \dots, n, n, 0, n)$$
$$cr(v_1/W) = (n + 2, n, n, n, \dots, n, n, n, n)$$
$$cr(v_2/W) = (n, n, n, n, \dots, n, n, n, 0)$$
$$cr(v_3/W) = (n, n + 2, n, n, \dots, n, n, n, n)$$
$$cr(v_4/W) = (n, n, n + 2, n, \dots, n, n, n, n)$$

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$$cr(v_{n-1}/W) = (n, n, n, n, \dots, n, n + 2, n, n)$$
$$cr(v_n/W) = (n, n, n, n, \dots, n, n, n + 2, n)$$

Since $cr(v/W)$ are distinct for all $v \in V(H_{n,n})$, it follows that W is a circular resolving set of G. Since $G[W]$ is connected, W is a connected circular resolving set of G. Therefore $cdim_c(G) \leq n$. We substantiate that $cdim_c(G) = n$. Consider, however, that $cdim_c(G) \leq n - 1$. Then, a set

W'exists such that $|W'| \leq n-1$. As a result, there are at least two vertices u, v that satisfy the contradiction

$$cr(u/S') = cr(v/S') = (n, n, n, n, \dots, n, n).$$

Consequently, $cdim_c(G) = n$.

3.Some results on connected circular metric dimension of a graph

Theorem 3.1. For connected graph of order $n \geq 2$, $1 \leq cdim(G) \leq cdim_c(G) \leq n-1$.

Proof: Any circular resolving set of G needs atleast one vertex and so $cdim(G) \geq 1$. Since any connected circular resolving set is also a circular resolving set of G, we have $cdim(G) \leq cdim_c(G)$. Also since $V(G) - x$ is a connected resolving set of G, where $x \in V(G)$ is not a cut vertex of G, we have $cdim_c(G) \leq n-1$. Thus $1 \leq cdim(G) \leq cdim_c(G) \leq n-1$.

Remark 3.2. The bounding bound in Theorem 3.1 bounds are sharp.

For $G = P_n$, $n \geq 2$, by theorem 2.5 $cdim_c(G) = 1$.

For the cycle $G = C_4$, by theorem 2.6 $cdim_c(G) = 3$ and for $G = K_n$, $n \geq 3$, $cdim_c(G) = n-1$.

Remark 3.3. Also, the bounds in Theorem 3.1 can be strict. For the star $G = K_{1,4}$, $cdim(G) = 3$ $cdim_c(G) = 4$ and $n = 5$. Thus $1 < cdim(G) < cdim_c(G) < n-1$.

Theorem 3.4. Let G be a connected graph of order $n \geq 3$ has connected circular metric dimension 1 if and only if $G = P_n$.

Proof. Let $G = P_n$. Then the result follows from Theorem 2.5. Conversely, assume that $cdim_c(G) = 1$. Let $W = \{v\}$ be a minimum connected circular resolving set of G. Then $cr(u/W) = D^c(u, v)$ is a non-negative integer less than $2(n-1)$ for each $u \in V(G)$. There exists a vertex $u \in V(G)$ such that $d(u, v) = n-1$. This is because the representation of $V(G)$ with regard to W are distinct. As a result, the circular diameter of G is $2(n-1)$, implies that $G = P_n$.

Theorem 3.5. Let G be a connected graph of order $n \geq 3$. If every pair of vertices of G is a circular diametral path of G. Then $cdim_c(G) = n-1$

Proof: Assume that every pair of vertices of G is circular diametral path of G. Therefore $D^c(u,v) = n$ for all $u, v \in V(G)$. Hence it follows that every circular resolving set of G contains at least n-1 elements. Also, $G[W]$ is connected. Hence $cdim_c(G) = n - 1$.

Remark 3.6. The converse of the Theorem 3.5. need not be true. For the graph $G = K_{1,n-1}$, $cdim_c(G) = n - 1$. But there are at least two vertices say x and y in G such that $x - y$ is not a circular diametral path of G.

Theorem 3.7. For any pair of integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $cdim(G) = a$ and $cdim_c(G) = b$.

Proof: For $a = b$, let $G = K_{a+1}$ then by Theorems 1.1 and 2.7, $cdim(G) = cdim_c(G) = a$. So let $2 \leq a < b$. Let P_{b-a} be a path of order b-a+1 and let $V(P_{b-a+1})= \{v_1, v_2, \ldots, v_{b-a+1}\}$. Let G be the graph obtained from by adding the new vertices $u_1, u_2, \ldots, u_a$ and introducing the edge $v_{b-a+1}u_i$ $(1 \leq i \leq a)$. The graph is shown in Figure 3.1.

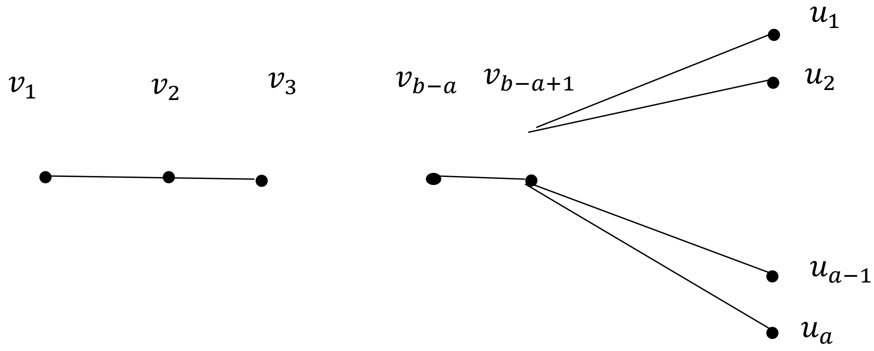

Figure 3.1

First, we prove that $cdim(G) = a$. Let $Z = \{u_1, u_2, \ldots, u_a\}$. Then every circular resolving set of G contains atleast $a - 1$ vertices from Z and the vertex v_1 and so $cdim(G) \geq a - 1 + 1 = a$. Let $S = Z \cup \{v_1\}$. Then S is a circular resolving set of G so that $cdim(G) = a$. Next, we prove that $cdim_c(G) = b$. By Observation 2.4(ii), $Z_1 = \{v_2, v_3, \ldots, v_{b-a+1}\}$ is a subset of every connected circular resolving set of G. Also it is easily seen that every connected circular resolving set of G contains atleast a-1 vertices from Z and the vertex v_1 and so $cdim_c(G) = b - a + 1 + a - 1 = b$. Let $S_1 = S \cup Z_1$. Then S_1 is a connected circular resolving set of G so that $cdim_c(G) = b$.

Conclusion

This article established a novel circular distance metric called the connected circular metric dimension in graphs. We will develop this concept to incorporate more distance considerations in a subsequent investigation.

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