Every bipartite graph is an induced subgraph of a sum graph

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Abstract

A finite simple graph G is called an integral sum graph(respectively, sum graph) if there is a bijection f from the vertices of G to a set of integers S (respectively, a set of positive integers S) such that uv is an edge of G if and only if $f(u)+f(v) \in S$. In 1999, Liaw et al (Ars Comb.,Vol.54, 259-268) posed the conjecture that every tree is an integral sum graph. In this note, we prove that every bipartite graph is an induced subgraph of a sum graph G with sum number $\sigma(G) = 1$.

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1 Introduction

All the graphs considered in this paper are finite simple graphs. Terms that are not defined here can be referred from the book [10]. Sum Graphs and Integral Sum Graphs were introduced by Harary [5]. A graph G is called a sum graph if the vertices of G can be labeled with distinct positive integers so that $e = uv$ is an edge of G if and only if the sum of the labels of the vertex u and vertex v is also a label in G. It is clear that if G is a properly labeled sum graph, then the vertex receiving the highest label can not be adjacent to any other vertex. Thus, every sum graph must contain isolated vertices. In otherwords, a connected graph is not a sum graph. If G is not a sum graph, adding a finite number of isolated vertices to it always yields a sum graph. Given any graph G with p vertices and q edges, it is trivial that the union $G \cup qK_1$ of G with q isolated vertices is a sum graph. We can define the sum number $\sigma(G)$ of G as the smallest number

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(say s) of isolated vertices added to G such that $G \cup sK_1$ is a sum graph.

An integral sum graph is also defined just as the sum graph, difference being that the label set S is a subset of Z, the set of integers. The integral sum number $\zeta(G)$ is the smallest non-negative integer s such that $G \cup sK_1$ is an integral sum graph. Clearly for any graph G, $\zeta(G) \leq \sigma(G)$. For a survey on sum graphs and integral sum graphs, we refer to the dynamic survey on graph labeling by Gallian [4].

Liaw et al [7] posed the conjecture that every tree is an integral sum graph. This conjecture was proved only for some classes of trees: caterpillars, banana trees, generalized stars and trees whose forks(by fork we mean a vertex of degree not 2) are distance at least 4 from each other [1, 2]. He et al [6] reduced this distance to 3. Pyatkin [8] proved that every tree whose forks are at least distance 2 apart is an Integral Sum Graph. Also, Pyatkin proved that subdivided trees are integral sum graphs. Ellingham [3] proved that $\sigma(T) = 1$ for every $T \neq K_1$. Tiwari and Tripathi [9] gave some bounds on the number of edges for a graph to be sum graphs and integral sum graphs. In this paper, we prove a characterization result that every bipartite graph is an induced subgraph of a sum graph G with sum number $\sigma(G) = 1$.

2 Characterization of Sum Graphs

In this section, we prove that any bipartite graph is an induced subgraph of a sum graph G with sum number $\sigma(G) = 1$.

Theorem 1. *Let $B = (V_1, V_2)$ be any bipartite graph with $|V_1| \geq |V_2|$. Then there exists a sum graph G with $\sigma(G) = 1$ such that B is an induced subgraph of G.*

Proof. Given that $B = (V_1, V_2)$ is a bipartite graph with $|V_1| \geq |V_2|$. Let $|V_1| = r$ and $|V_2| = s$. Consider the vertices in V_1 as $\{u_1, u_2, \cdots, u_r\}$ and the vertices in V_2 as $\{v_1, v_2, \cdots, v_s\}$. Define the vertex labeling function f as $f(u_i) = 2i - 1$, for $1 \leq i \leq r$ and $f(v_i) = 2i$, for $1 \leq i \leq s$. By the definition of f, it is clear that f is one-to-one. Now, define the edge label for an edge $e = uv$ as $f(e) = f(u) + f(v)$. Since B is a bipartite graph, every edge in B has one end in V_1 and the other end in V_2, and being all the vertex labels of vertices in V_1 are odd whereas vertex labels of vertices in V_2 are even, by the definition of the edge labels, the edge label of any edge in B is an odd number. Define V_f = {the set of all vertex labels of vertices of B} and E_f =

$\{$the set of all edge labels of edges of $B\}$. Let $L = E - (V_f \cap E_f) = \{x_1, x_2, \cdots, x_t\}$ and let z be the maximum label among the labels in L.

Now, let us construct the bipartite graph G as follows: Start with the bipartite graph B along with their labels as defined by the function f. Add an isolated vertex with label z. Define $L' = L - \{z\}$. Until $L' = \phi$, choose a label(say y) from L', add a vertex with vertex label y to the vertex with vertex label $z - y$ and remove the label y from the set L'. Observe that by the construction of graph G, G is a sum graph with one isolated vertex. Therefore, $\sigma(G) = 1$. Thus, we have constructed a sum graph G with $\sigma(G) = 1$in such a way that given bipartite graph B is an induced subgraph of G. Hence the proof. □

3 Illustrative Example

For the bipartite graph in Figure 4, the corresponding sum graph G with $\sigma(G) = 1$ is shown in Figure 5.

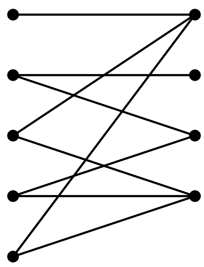

Figure 1: Bipartite Graph $B(V_1, V_2)$

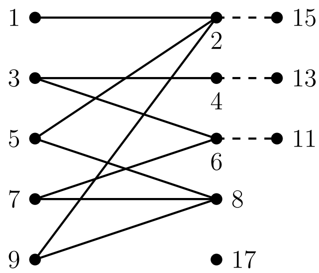

Figure 2: Sum Graph G with bipartite graph B as an induced subgraph

4 Conclusion

In this note, we proved a characterization result that any bipartite graph is an induced subgraph of a sum graph G with $\sigma(G) = 1$. We have given a constructive procedure on to generate such sum graphs G for a given bipartite graph.

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