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On the circular metric dimension of a graph

S. Sheeja
Research Scholar, Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies, Chennai, Tamil Nadu, India

K. Rajendran
Associate Professor, Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies, Chennai, Tamil Nadu, India

ABSTRACT: Let $G = (V, E)$ be a simple graph. Let u, v be any two vertices of G. The circular distance between u and v denoted by $D^c(u, v)$ and is defined by

$$D^c(u,v) = \begin{cases} D(u,v) + d(u,v) & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

where $D(u, v)$ and $d(u, v)$ are detour distance and distance between u and v respectively. Let $W = \{w_1, w_2, \dots, w_k\} \subset V(G)$ and $v \in V(G)$. The representation $cr(v/W)$ of v with respect to W is the k-tuple $(D^c(v, w_1), D^c(v, w_2), \dots, D^c(v, w_k))$. If various vertices of G have distinct representations with regard to W, then W is referred to as a circular resolving set. For each given G, a circular resolving set of minimum cardinality is referred to as a *cdim* -set. The circular metric dimension of G, denoted by $cdim(G)$, is the cardinality of the *cdim* -set. A few general qualities that this idea satisfies are examined. A few common graphs' circular metric dimensions are found. We characterise connected graphs of order $n \geq 2$ with dimension 1 in the circular metric. It is shown that for every pair of integers a nd n with $1 \leq a \leq n - 1$, there exists a connected graph of order n such that $cdim(G) = a$. The circular metric dimension for the total graph of paths, the middle graph of paths are determined. Additionally, the circular metric dimension for the corona products of some graphs are determined.

Keywords: detour distance, distance, circular resolving set, circular distance, circular metric dimension, AMS Subject Classification: 05C12

1 INTRODUCTION

Let G be a simple graph having an edge set $E(G)$ and a vertex set $V(G)$. A graph G has an order of $|V(G)|$, with n being the number of vertices. A graph G has a length of $|E(G)|$ and m edges, which represent its number of edges. We refer to [1] for a basic terminology of graph theory. The number of edges incident to a vertex $v \in V(G)$ is its degree $\deg(v)$. The greatest degree of a graph G is shown by $\Delta(G)$. The length of the shortest path between two vertices $u, v \in V(G)$ is the distance $d(u, v)$. The detour distance $D(u, v)$ between two vertices $u, v \in V(G)$ is the length of a longest path between them. These concepts were studied in [2]. The circular distance between u and v is denoted by $D^c(u, v)$ and is defined by

$$D^c(u,v) = \begin{cases} D(u,v) + d(u,v) & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

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The circular distance has a significant impact on logistical management. As an illustration, consider a milk van that travels far to deliver milk from a dairy to the final location covering each delivery location along the way in the shipment of milk to a from a dairy in town “*A*” a location in the town of “*B*”. The return journey in order to limit the amount of time, gasoline, and money spent on the vehicle while the quickest path to deliver milk could be chosen attain the dairy. These concepts were studied in [3]. The metric dimension of a graph was studied in [4–6]. The circular metric dimension of a graph is a novel metric dimension that we examine in this paper.

2 THE CIRCULAR METRIC DIMENSION OF A GRAPH

2.1 *Definition*

Let $W = \{w_1, w_2, \ldots, w_k\} \subset V(G)$ be an ordered set and $v \in V(G)$. The circular representation $cr\left(\frac{v}{W}\right)$ of v with respect to W is the k-tuple $(D^c(v, w_1), D^c(v, w_2), \ldots, D^c(v, w_k))$. Numerous vertices of G have distinct representations with regard to W, then W is referred to as a circular resolving set. A cdim -set of G is a circular resolving set of minimum number of elements; the circular metric dimension of G, denoted by $cdim(G)$, is the cardinality of a cdim -set of G.

2.2 *Example*

Let $W = \{v_1, v_2\}$ for the graph G shown in Figure 1. Consequently $cr(v_1/W) = (0, 4)$, $cr(v_2/W) = (4, 0), cr(v_3/W) = (3, 4)$, $cr(v_4/W) = (4, 5)$. W is a circular resolving set of G as a result $cdim(G) \leq 2$. $cdim(G) = 2$ because $V(G)$ does not have a singleton subset that is a circular resolving set of G.

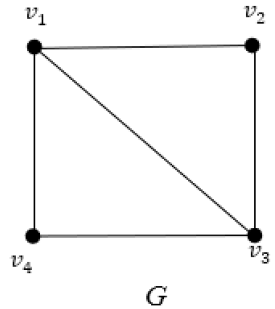

Figure 1. Circular resolving set of G.

2.3 *Observation*

Let G be a connected graph of order $n \geq 2$. Then $1 \leq cdim(G) \leq n - 1$.

In the following section, we calculate the circular metric dimension of certain standard graphs.

2.4 *Theorem*

For the graph $G = P_n$ $(n \geq 2)$, $cdim(G) = 1$.

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $W = \{v_1\}$. Consequently $D^c(v_1, v_i) = 2(i - 1)$, $(1 \leq i \leq n)$. Since $cr(v/W)$ is distinct for all $v \in V(P_n)$, it follows that W is a circular resolving set of G. As a result $cdim(G) = 1$.

2.5 *Theorem*

For the cycle $G = C_n$, $n \geq 3, cdim(G) = n - 1$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $W = \{v_1, v_2, \ldots, v_{n-1}\}$, Then the circular metric representations $(n - 1)$ tuples are as follows

$$cr(v_1/W) = (0,\ n, n, \ldots, n),$$

$$cr(v_2/W) = (n, 0,\ n, n, \ldots, n)$$

$$cr(v_3/W) = (n, n, 0,\ n, n, \ldots, n)$$

$$cr(v_{n-1}/W) = (n, n, n, \ldots,\ n, 0)$$

$cr(v_n/W) = (n, n, n, n, \ldots, n)$. Since the circular metric representation are distinct, $cdim(G) \leq n - 1$. We substantiate that $cdim(G) = n - 1$. Consider, however, that $cdim(G) \leq n - 2$. After that, there is a set S' such that $|S'| \leq n - 2$. Consequently, the contradiction is satisfied by at least two vertices $(u,\ v)$.

$cr(u/S') = cr(v/S') = (n, n, \ldots, n)$. Consequently, $cdim(G) = n - 1$.

$|S'|$ tuples.

2.6 *Theorem*

For the complete graph $G = K_n$, $n \geq 2, cdim(G) = n - 1$.

Proof. The proof is analogous to Theorem 2.5.

2.7 *Theorem*

For the complete bipartite graph $G = K_{r,s}$, $(1 \leq r \leq s)$,

$$cdim(G) = \begin{cases} 1; r = 1,\ 1 \leq s \leq 2, \\ r + s - 2; r = 1,\ s \geq 3 \\ r + s - 1; 2 \leq r \leq s \end{cases}.$$

Proof. Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the two bipartite sets of G. We have the three cases.

Case (i): $r = 1,\ 1 \leq s \leq 2$. Theorem 2.4 provides the desired outcome.

Case (ii): $r = 1,\ s \geq 3$.

Let $W = V(G) - \{x_1, y_s\}$. Then, the $(n - 2)$ tuples representing the circular metric are as follows:

$$cr(x_1/W) = (2, 2, 2, \ldots, 2, 2)$$

$$cr(y_1/W) = (0, 4,\ 4, \ldots, 4, 4)$$

$$cr(y_2/W) = (4, 0, 4,\ \ldots, 4, 4)$$

$$cr(y_{s-1}/W) = (4, 4, 4, \ldots,\ 4, 0)$$

$$cr(y_s/W) = (4, 4, 4, \ldots, 4, 4).$$

As the representations are distinct, W is a set of circular resolves for G, so that $cdim(G) \leq r + s - 2$. We demonstrate that $cdim(G) = r + s - 2$. On the other hand, imagine that, $cdim(G) \leq r + s - 3$. Then there exists a circular resolving set S' such that. $|S'| \leq r + s - 3$. Consequently, at least two end vertices exist there exists $u, v \in V \backslash S'$ such that $cr(u/S') = cr(v/S') = (4, 4, 4, \ldots, 4, 4)$, which is incoherent. As a result, $cdim(G) = r + s - 2$.

Case (iii): $2 \leq r \leq s$.

Let $W = V(G) - \{y_s\}$. Then, the following are the $(r + s - 1)$ tuples that represent the circular metric:

$$cr(x_1/W) = (0, r + s - 1, r + s - 1, \ldots, r + s - 1)$$

$$cr(x_2/W) = (r + s - 1, 0, r + s - 1, \ldots, r + s - 1)$$

$$cr(x_r/W) = (r + s - 1, r + s - 1, \ldots, 0, r + s - 1, \ldots, r + s - 1)$$

r^{th} place

$$cr(y_1/W) = (r + s - 1, r + s - 1, \ldots, r + s - 1, 0, r + s - 1 \ldots, r + s - 1)$$

$(r + 1)^{\text{th}}$ place

$$cr(y_2/W) = (r + s - 1, r + s - 1, \ldots, r + s - 1, 0,\ r + s - 1\ \ldots, r + s - 1)$$

$(r + 2)^{\text{th}}$ place

$$cr(y_{s-1}/W) = (r + s - 1, r + s - 1, r + s - 1, \ldots, r + s - 1, r + s - 1, \ldots, 0)$$

$(r + s - 1)^{\text{th}}$ place

$$cr(y_s/W) = (r + s - 1, r + s - 1, r + s - 1, \ldots, r + s - 1, r + s - 1, \ldots, r + s - 1).$$

Since the representation are distinct, W is a circular resolving set of G so that $cdim(G) \leq r + s - 1$. We demonstrate that $cdim(G) = r + s - 1$. Consider however, that $cdim(G) \leq r + s - 2$. If so, a circular resolving set S' exists such that $|S'| \leq r + s - 2$ As a result, there are at least two vertices, $u, v \in V \backslash S'$such that $cr(u/S') = cr(v/S') = (r + s - 1, r + s - 1, r + s - 1, \ldots, r + s - 1)$, which is incoherent. As a result, $cdim(G) = r + s - 1$.

2.8 *Theorem*

Let G be a connected graph of order $n \geq 2$ has circular metric dimension 1 if and only if $G = P_n$.

Proof. Let $G = P_n$. Then the result follows from Theorem 2.5

Conversely, assume that $cdim(G) = 1$. Let $W = \{v\}$ be a minimum circular resolving set of G. Then $cr(u/W) = D^c(u, v)$ is a non-negative integer less than $2(n - 1)$ for each $u \in V(G)$. There exists a vertex $u \in V(G)$ such that $d(u, v) = n - 1$. This is because the representation of $V(G)$ with regard to W are distinct. As a result, the diameter of G is $2(n - 1)$, implies that $G = P_n$.

3 THE CIRCULAR METRIC DIMENSION OF SOME SPECIAL GRAPHS

A graph's total graph and middle graph are examined in [2]. In [6], the corona product of graphs is also explored. In this section, the circular metric dimension for the total graph of paths, the middle graph of paths are determined. Additionally, the circular metric dimension for the corona products of some graphs are determined.

3.1 *Theorem*

For the graph $G = C_n \circ K_1$, $(n \geq 4), cdim(G) = n$.

Proof. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ be the set of end vertices of G. Let $W = \{v_1, v_2, \ldots, v_n\}$. Then the circular metric representations n tuples are as follows

$$cr(v_1/W) = (0, n+4, n+4, \ldots, n+4)$$

$$cr(v_2/W) = (\ n+4, 0,\ n+4, \ldots, n+4)$$

$$cr(v_n/W) = (n+4, n+4, \ldots n+4, 0)$$

$$cr(u_1/W) = (2, n+2, n+2, n+2, \ldots,\ n+2)$$

$$cr(u_2/W) = (n+2, 2, n+2, \ldots,\ n+2)$$

$$cr(u_3/W) = (n+2, n+2,\ 2, \ldots,\ n+2)$$

$$cr(u_n/W) = (n+2, n+2, \ldots, n+2,\ 2).$$

W is a circular resolving set of G since the representations are distinct and as a result, $cdim(G) \leq n$. We establish that $cdim(G) = n$. Consider, however, that $cdim(G) \leq n-1$. If so, a circular resolving set S' exists such that $|S'| \leq n-1$. Therefore, either two end vertices or at least two cut vertices of G belongs to $V \backslash S'$. Allow $u, v \in V/S'$. If G's end vertices are u and v, then $cr(u/S') = cr(v/S') = (n+2, n+4, n+4, \ldots, n+4)$.

If G's cut vertices are u and v, then $cr(u/S') = cr(v/S') = (n, n+2, n+2, \ldots, n+2)$.

Which is incongruous. Consequently, $cdim(G) = n$.

3.2 *Theorem*

For the graph $G = K_n \circ K_1$, $(n \geq 4), cdim(G) = n$.

Proof. The proof is similar to the Theorem 3.1.

3.3 *Theorem*

Let G be the middle graph of the path $P_n (n \geq 3)$. Then $cdim(G) = 2$.

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $u_i = v_i v_{i+1}$, $1 \ \leq i \leq n-1$. Then $V(G) = \{v_i, u_j,\}(1 \leq i \leq n,\ 1 \leq j \leq n-1)$. In G, v_i is adjacent to u_1, v_n is adjacent to u_{n-1}. v_i is adjacent to u_{i-1}, u_i, $2 \leq i \leq n-1$, u_1 is adjacent to v_1, v_2 and u_2, u_i is adjacent to u_{i-1}, u_{i+1}, v_i and v_{i+1} for$2 \leq i \leq n-2$ and u_{n-1} is adjacent to $v_{n-2}v_{n-1}$ and v_n.

Evidently, $|V(G)| = 2n-1$. As a result of $G \neq P_n$, by Theorem 2.8, $cdim(G) \geq 2$. Let $W = \{v_1, v_n\}$, $Cr(v_1/W) = (0,\ 3n-2)$, $Cr(v_2/W) = (5, 3n-4)$, $\ldots, Cr(v_{n-1}/W) = (3n-5, 5), Cr(v_n/W) = (3n-2,\ 0)$, $Cr(u_1/W) = (2,\ 3n-4), Cr(u_2/W) = (5, 3n-7)$, $Cr(u_3/W) = (8, 3n-10), ..., Cr(u_{n-1}/W) = (3n-4,\ 2)$. W is a circular resolving set of G, and since the circular metric representation is distinct, $cdim(G) = 2$.

3.4 *Theorem*

Let G be the total graph of the path $P_n (n \geq 4)$. Then $cdim(G) = 2$.

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $u_i = v_i v_{i+1}$, $1 \ \leq i \leq n-1$. Then $V(G) = \{v_i, u_j, 1 \leq i \leq n,\ 1 \leq j \leq n-1\}$. In G, v_1 is adjacent to v_2 and u_1;v_n is adjacent to v_{n-1} and u_{n-1};v_i is adjacent to v_{i-1}, v_{i+1}, u_{i-1} and u_i; $2 \leq i \leq n-1$, u_1 is adjacent to v_1,v_2 and u_2, u_i is adjacent to u_{i-1}, u_{i+1}, v_i and v_{i+1} for $2 \leq i \leq n-2$ and u_{n-1} is adjacent to v_{n-2}, v_n and u_{n-2}. Evidently, $|V(G)| = 2n-1$. As a result of $G \neq P_n$, by Theorem 2.8, $cdim(G) \geq 2$. Let

$W = \{v_1, v_n\}$, $Cr(v_1/W) = (0, 3n-3), Cr(v_2/W) = (2n-1, 3n-4), Cr(v_3/W) = (2n, 3n-5), \ldots, Cr(v_{n-1}/W) = (3n-4, 2n-1), Cr(v_n/W) = (3n-3, 0), Cr(u_1/W) = (2n-1, 3n-3), Cr(u_2/W) = (2n, 3n-4), Cr(u_3/W) = (2n+1, 3n-5), \ldots,$

$Cr(u_{n-1}/W) = (3n-4, 2n), Cr(u_n/W) = (3n-3, 2n-1)$. W is a circular resolving set of G, and since the circular metric representation is distinct, $cdim(G) = 2$.

3.5 *Theorem*

Let G be the graph obtained from $K_{1,n-1}$, $(n \geq 4)$, by subdividing the end edges exactly once. Then $cdim(G) = n-2$.

Proof. Let x be the central vertex of $K_{1,n-1}$ $(n \geq 4)$ and the set of end vertices of $K_{1,n-1}$ $(n \geq 4)$ be $\{v_1, v_2, \ldots, v_{n-1}\}$. G is the graph created by dividing xv_i $(1 \leq i \leq n-1)$ by u_i $(1 \leq i \leq n-1)$ from $K_{1,n-1}$, $(n \geq 4)$. Let $W = (u_1, u_2, \ldots, u_{n-3}, v_{n-2}\}$. Then the circular metric representations $(n-2)$ tuples are as follows

$$cr(x/W) = (2, 2, \ldots, 2, 4)$$

$$cr(u_1/W) = (0, 4, 4, \ldots, 4, 6)$$

$$cr(u_2/W) = (4, 0, 4, \ldots, 4, 6)$$

$$cr(u_{n-3}/W) = (4, 4, 4, \ \ldots, 0, 6)$$

$$cr(u_{n-2}/W) = (4, 4, 4, \ldots, 4, 2)$$

$$cr(u_{n-1}/W) = (4, 4, \ldots, 4, 8)$$

$$cr(v_1/W) = (2, 6, 6, 6, \ldots, 6, 8)$$

$$cr(v_2/W) = (6, 2, 6, 6, \ldots, 6, 8)$$

$$cr(v_{n-3}/W) = (6, 6, \ldots, 2, 8)$$

$$cr(v_{n-2}/W) = (6, 6, 6, \ldots, 6, 0)$$

$$cr(v_{n-1}/W) = (6, 6, 6, \ldots, 6, 8).$$

Due to the distinctness of the representations, W is a circular resolving set of G such that $cdim(G) \leq n-2$. We substantiate that $cdim(G) = n-2$. Consider, however, that $cdim(G) \leq n-3$. If so, a circular resolving set S' exists such that $|S'| \leq n-3$. Because of this, there are at least two vertices, $u, v \in V \backslash S'$such that $cr(u/S') = cr(v/S') = (6, 6, 6, \ldots, 6, 8)$, which is incongruous. Consequently, $cdim(G) = n-2$.

4 CONCLUSION

In the present research, the circular dimension in graphs—a novel distance metric—was established. In a follow-up study, we will refine this idea to include other distance concerns.

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