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The Forcing Circular Number of a Graph

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Abstract:
Let S be a cr-set of graph G and let G be a connected graph. If S is the only cr-set that contains T, then a subset $T \subseteq S$ is referred to be a forcing subset for S. A minimum forcing subset of S is a forcing subset for S of minimum cardinality. The cardinality of a minimum forcing subset of S is the forcing circular number of S, represented by the notation $f_{cr}(S)$. $f_{cr}(G)$ = min $\{f_{cr}(S)\}$ is the forcing circular number of G, where the minimum is the sum of all minimum forcing circular-sets S in G. For several standard graphs, the forcing circular number is identified. It is demonstrated that there exists a connected graph G such that $f_g(G) = a$ and $f_{cr}(G) = b$ for every integer $a \geq 0$, and $b \geq 0$.

Keywords: cr-set, circular number, forcing circular number.

AMS Subject Classification: 05C12.

1. Introduction and Preliminaries

A graph $G = (V, E)$ is a connected, finite graph that does not have loops or numerous edges. G is represented by the symbols n and m, respectively, for order and size. We use [1,6] for basic terminology in graph theoretic. If $uv \in E(G)$, then two vertices, u and v, are considered nearby in G. The collection of vertices next to a vertex v in G is called its neighbourhood, or $N(v)$. The vertex v has a degree of $deg(v) = |N(v)|$. We refer to u as an end edge, u as a leaf, and v as a support vertex if $e = \{u, v\}$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$. The greatest degree of a graph G is shown by $\Delta(G)$. $G[S]$ is the representation of the subgraph that a set S of vertices of a graph G induces, where $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) : u, v \in S\}$. A vertex v is an extreme vertex of G if and only if $G[N(v)]$ is complete.

The length of the shortest path between two vertices $u, v \in V(G)$ is the distance $d(u, v)$. A $u - v$ geodesic of G is any $u - v$ path of length $d(u, v)$. If x is a vertex of P and $x \neq u, v$, then x is an internal vertex of a $u - v$ path P. $I[u, v]$ is the closed interval consisting of u, v and all vertices that are on a $u - v$ geodesic of G. The closure of a non-empty set $S \subseteq V(G)$ is given by the set $I[S] = \cup_{u,v \in S} I[u, v]$. If $I[S] = V(G)$, then a set $S \subseteq V(G)$ is a geodetic set. The geodetic number of G, represented by $g(G)$, is the lowest cardinality of a geodetic set of G. A g −set of G is a geodetic set of minimum cardinality. See [3,4,8] for references on geodetic parameters in graphs.

The longest path between two vertices $u, v \in V(G)$ is the detour distance $D(u, v)$. A $u - v$ detour of G is any $u - v$ path of length $D(u, v)$. All vertices of the closed interval $I_D[u, v]$ lie on some $u - v$

detour of G, and the interval itself consists of u, v. The closure of a non-empty set $S \subseteq V(G)$ is given by the set $I_D[S] = \bigcup_{u,v\in S} I_D[u, v]$. A detour set is then defined as a set $S \subseteq V(G)$. The detour number of G, represented by $dn(G)$, is the lowest cardinality of a detour set of G. A dn-set of G is a diversion set with minimum cardinality. Hence [5,7] covered the study of these ideas.

$D^c(u, v)$ represents the circular distance between u and v, which is represented as

$$D^c(u, v) = \begin{cases} D(u, v) + d(u, v) & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

The detour distance and the distance between u and v are denoted by $D(u, v)$ and $d(u, v)$, respectively. The circular diameter D^C is the longest circular distance between 2 vertices on G. An $u - v$ circular of G is any $u - v$ path of length $D^c(u, v)$. The circular diameter D^C is the longest circular distance between 2 vertices on G. For $u, v \in V$, $I_c[u, v]$ represents group of every vertex positioned on a $u - v$ circular in G. For $S \subseteq V(G)$, let $I_c[S] = \bigcup_{u,v \in S} I_c[u, v]$. These concepts were studied in [2,9,10].

Theorem 1.1. [2] In a connected graph, every geodetic set of G has an extreme vertex.

Theorem 1.2. [2] Let W be the set of all geodetic sets in graph G. Then $f_g(G) \leq g(G) - |W|$.

2. The forcing circular number of a graph

Definition 2.1. A subset $T \subseteq S$ is referred to as a forcing subset for S, if S is the only cr-set that contains T. A forcing subset of minimum cardinality for S is known as a minimum forcing subset of S. The forcing circular number of G is denoted by the notation $f_{cr}(G) = \min\{f_{cr}(S)\}$, where the minimum is established over all cr-sets S in G. The cardinality of a minimum forcing subset of S is the forcing circular number of S.

Example 2.2. The only two cr-sets of the graph G displayed in Figure 2.1 are $S_1 = \{v_1, v_4, v_5\}$ and $S_2 = \{v_1, v_4, v_6\}$ such that $f_{cr}(S_1) = f_{cr}(S_2) = 1$ and $f_{cr}(G) = 1$.

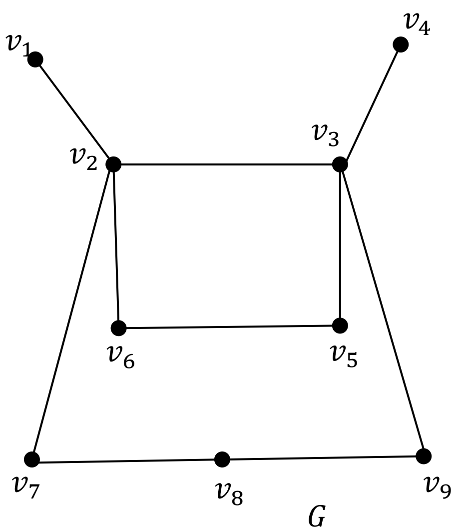

Figure 2.1

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Observation 2.3. For each graph G that is connected, $0 \leq f_{cr}(G) \leq cr(G)$.

Remark 2.4. Observation 2.3 has sharp bounds. For $G = P_3$, $f_{cr}(G) = 0$. For $G = C_4$ with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$, $S_1 = \{v_1, v_2\}$, $S_2 = \{v_2, v_3\}$, $S_3 = \{v_3, v_4\}$, $S_4 = \{v_7, v_4\}$, $S_5 = \{v_1, v_3\}$ and $S_6 = \{v_2, v_4\}$ are the only six cr-sets of G there exists $f_{cr}(S_i) = 2$, $1 \leq i \leq 6$ so that $f_{cr}(G) = cr(G) = 2$.

Additionally, the limitations in Observation 2.3 may be extremely rigorous. The graph G shown in Figure 2.1 has two values: $cr(G) = 2$ and $f_{cr}(G) = 1$. Hence $0 < f_{cr}(G) < cr(G)$. **Theorem 2.5.** Consider a connected graph, G. Following that

i) $f_{cr}(G) = 0$ iff G has a unique minimum cr-set of G.

ii) $f_{cr}(G) = 1$ iff G possesses a minimum of two cr-sets, at least one of which is a distinct cr-et that includes one of its elements.

iii) $f_{cr}(G) = cr(G)$ iff any proper subset of G that is not contained in any cr-set is the unique minimal cr-set of G.

Definition 2.6. A vertex v of a connected graph G. If v belongs to each cr-set of G, then $v(G)$ is considered to be a circular vertex of G.

Example 2.7. For the graph G shown in Figure 2.2, the set of all circular vertices of G is represented by $\{v_1, v_3, v_5\}$ since $S_1 = \{v_1, v_3, v_5, v_6\}$ and $S_2 = \{v_1, v_3, v_5, v_9\}$ are the only two cr-sets of G.

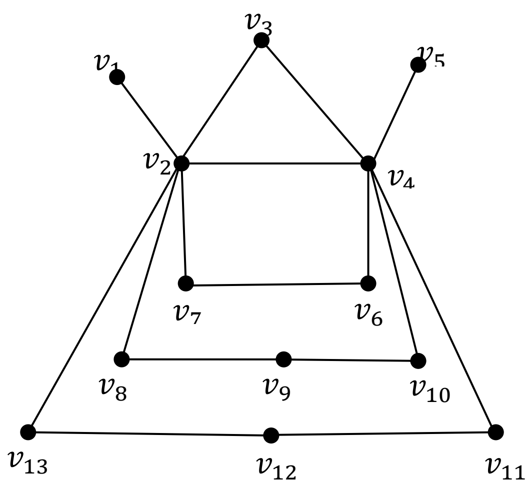

Figure 2.2

Theorem 2.8. Let W be the set of all circular vertices of connected graph G. Then $f_{cr}(G) \leq cr(G) - |W|$.

Remark 2.9. The bounds in Theorem 2.8 are precise. Regarding the graph G shown in Figure 2.2, $|W| = 3$, $cr(G) = 4$ and $f_{cr}(G) = 1$. Thus $f_{cr}(G) = cr(G) - |W|$. Moreover, the bounds in Theorem 2.8 may be rigid. With respect to graph G displayed in Figure 2.3, $S_1 = \{v_1, v_4, v_5, v_7\}$, $S_2 = \{v_1, v_4, v_5, v_8\}$, $S_3 = \{v_1, v_4, v_5, v_9\}$, $S_4 = \{v_2, v_4, v_5, v_7\}$, $S_5 = \{v_2, v_4, v_5, v_8\}$ and $S_6 =$

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$\{v_2, v_4, v_5, v_9\}$ are the six cr-sets of G so that $\{v_1, v_4, v_5\}$ is the set of all circular vertices of G there exists $f_{cr}(G) = 1$ and $cr(G) = 3$.

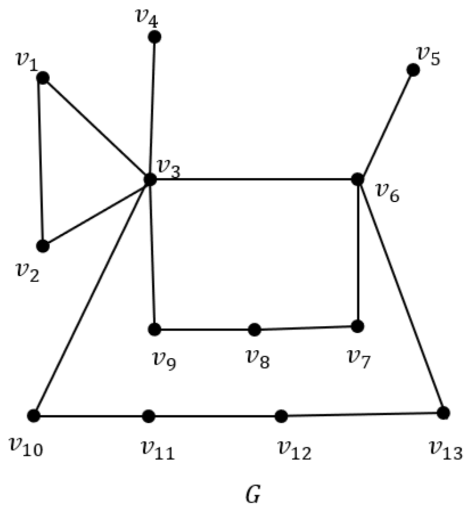

Figure 2.3

Theorem 2.10. For the complete bipartite graph $G = K_{r,s}, (1 \leq r \leq s)$,

$$f_{cr}(G) = \begin{cases} 0 & if\ r = 1, s \geq 2 \\ 2 & if\ 2 \leq r \leq s \end{cases}$$

Proof. Let $U = \{u_1, u_2, \dots, u_r\}$ and $W = \{w_1, w_2, \dots, w_s\}$ be the bipartite sets of G.

For $s \geq 2$ and $r = 1$, $S = W$ is the distinct cr-set of G so that $f_{cr}(G) = 0$. Hence $2 \leq r \leq s$. Let $w \in W$ and $u \in U$. Such that $S = \{u, w\}$ is a cr-set of G. Since this is true for all $u \in U$ and $w \in W$, S is not unique cr-set of G containing u or w. Therefore $f_{cr}(G) = 2$. As this holds true for every cr-sets S of G, $f_{cr}(G) = 2$.

Theorem 2.11. For the non-trivial tree T, $f_{cr}(T) = 0$.

Proof. Considering S to be the collection of all end vertices in G, S is the only cr-set in G such that $f_{cr}(G) = 0$.

Theorem 2.12. For the cycle $G = C_n, (n \geq 4)$, $f_{cr}(G) = 2$.

Proof. Let x and y represent any two vertices of G. There exists $S = \{x, y\}$ is a cr-set of G. Hence x and y are arbitrary, S is not a unique cr-set containing x or y. Therefore $f_{cr}(G) = 2$. As this holds true for all cr-sets S of G therefore $f_{cr}(G) = 2$.

Theorem 2.13. For the wheel $G = K_1 + C_{n-1}, (n \geq 5)$, $f_{cr}(G) = 1$.

Proof. Assume that x represents the central vertex of G and C_{n-1} be $v_1, v_2, \dots, v_{n-1}, v_1$. Then $S_i = \{x, v_i\}$ $(1 \leq i \leq n-1)$ and $S = \{u, v\}$ where u and v are any two vertices in C_{n-1} are the cr-sets of G. Now $f_{cr}(S_i) = 1$ $(1 \leq i \leq n-1)$. Since u and v are arbitrary, S is not a dsitinct cr-set containing u or v. Therefore $f_{cr}(G) = 2$. Hence it follows that $f_{cr}(G) = 1$.

Theorem 2.14. For the fan graph $F_n = K_1 + P_{n-1}, (n \geq 5)$, $f_{cr}(G) = 1$.

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Proof. Suppose that x represents the central vertex of of G and $V(P_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$. Then $S_i = \{x, v_i\}$ $(1 \le i \le n-1)$ and $S = \{u, v\}$ where u and v are any two vertices in P_{n-1} are the cr-sets of G. Now $f_{cr}(S_i) = 1$ $(1 \le i \le n-1)$. Since u and v are arbitrary, S is not a unique cr-set containing u or v. Therefore $f_{cr}(G) = 2$. Hence it follows that $f_{cr}(G) = 1$.

3. The Forcing Geodetic Numbers and the Forcing Circular Number of a Graph

The forcing geodetic numbers and the forcing circular number of a graph have no relationship, as the example below demonstrates.

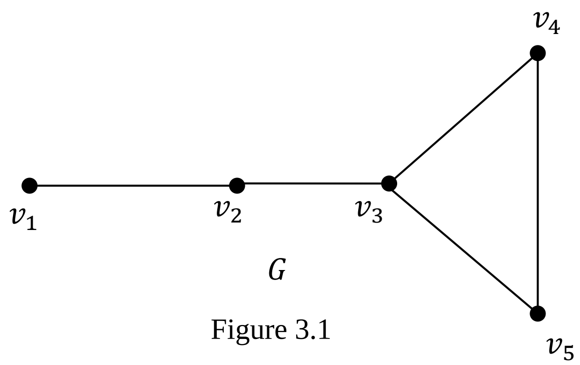

Figure 3.1

Example 3.1. The unique g-set of the graph G shown in Figure 3.1 is indicated as , $S = \{v_1, v_4, v_5\}$. Therefore $f_g(G) = 0$. Also $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_1, v_5\}$ are the only two cr-sets of G such that $f_{cr}(G) = 1$. Thus $f_g(G) < f_{cr}(G)$.

Example 3.2. The unique cr-set of the graph G shown in Figure 3.2 is represented as $S = \{v_1, v_2\}$. Therefore $f_{cr}(G) = 0$. Also $S_1 = \{v_1, v_3, v_6\}$ and $S_2 = \{v_1, v_4, v_6\}$ are the only g-sets of G so that $f_g(G) = 1$. Thus $f_g(G) > f_{cr}(G)$.

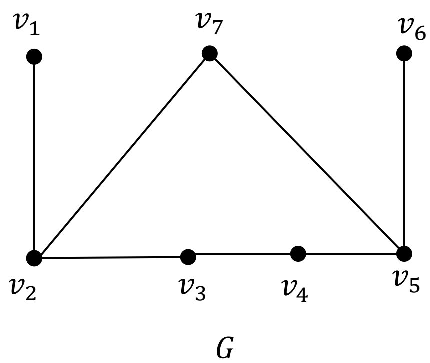

Figure 3.2

Theorem 3.3. In a connected graph G, $f_g(G) = a$ and $f_{cr}(G) = 0$ exist for each integer $a \ge 0$.

Proof. Assume that $P: u, v, w, x$ is an order four path. Consider $P_i: u_i, v_i$ $(1 \le i \le a)$ represent an identical pair of vertices. Let G be the graph generated by adding the edges vu_i and wv_i to P and P_i $(1 \le i \le a)$. The figure 3.3 displays the graph G.

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We first establish that $f_g(G) = a$. Let $Z = \{u, x\}$ represent all of G's end vertices. Z is a subset of every g-set in G, according to Theorem 1.1. For $(1 \leq i \leq a)$, consider $H_i = \{u_i, v_i\}$. It is easily shown that $g(G) \geq a$ since every vertex in the g-set of G contains exactly one vertex from each $H_i (1 \leq i \leq a)$. Let $S = Z \cup \{u_1, u_2, \dots, u_a\}$. As a result, S is a g-set of G and $g(G) = a + 2$, as $I[S] = V(G)$. For every g-set of G contains a subset, Z. By Theorem 1.2, $f_g(G) \leq g(G) - |Z| = a + 2 - 2 = a$. Therefore $f_g(G) \leq a$.

Considering that $g(G) = a + 2$ and that Z exists in every g-set of G, following that each g-set of G, if so, has the form $S = Z \cup \{c_1, c_2, \dots, c_a\}$, where $c_i \in H_i (1 \leq i \leq a)$. Given $|T| < a$, let T be any proper subset of S. After that, c_j $(1 \leq j \leq a)$ is a vertex such that $c_j \notin T$. Assume that b_j, a vertex of H_j, is distinct from c_j. Consequently, $S_1 = (S - \{c_j\}) \cup \{b_j\}$ is a g-set that properly contains T. As a result, T is not a forcing subset of S. For every minimum g-set of G, this holds true. Therefore $f_g(G) = a$.

Next we prove that $f_{cr}(G) = 0$. Since Z is the distinct cr-set of G, $f_{cr}(G) = 0$.

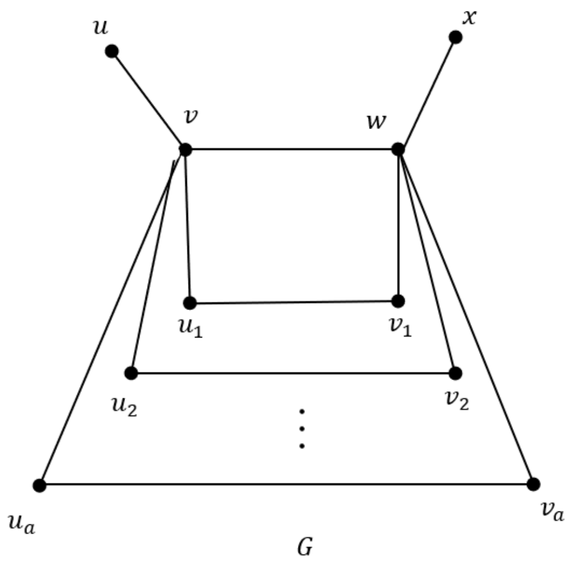

Figure 3.3

Theorem 3.4. For every integer $a \geq 0$, there exists a connected graph G such that $f_g(G) = 0$ and $f_{cr}(G) = a$.

For every integer $a \geq 0$, there exists a connected graph G such that $f_g(G) = a$ and $f_{cr}(G) = a$.

Proof. Let P': w_1, w_2, w_3 be a path of order 3, and consider P_i: t_1, t_2, t_3, t_4, t_5 be a path of order 5. Let P_i: r_i, s_i $(1 \leq i \leq a)$ be an order 2 replica of the path. Let G be the graph created by adding the edges $t_2 w_1, t_2 w_2, t_4 w_2, t_4 w_3, t_2 r_i$ $(1 \leq i \leq a)$ and $t_4 S_i$ $(1 \leq i \leq a)$ to P' and P_i $(1 \leq i \leq a)$. The Figure 3.4 displays the graph G.

First, we establish that $f_{cr}(G) = a$. Let the set of all of G's end vertices be $Z = \{t_1, t_5\}$. Such that Z is therefore a subset of each cr-set of G according to Theorem 1.1. H_i: $\{r_i, s_i\}$ $(1 \leq i \leq a)$ be given. Then, it is evident that $cr(G) \geq a + 2$ since every circular set of G has at least one vertex from each H_i $(1 \leq i \leq a)$. Let $S = Z \cup \{r_1, r_2, \dots, r_a\}$. $I_{D^c}[S] = V(G)$ in this case, indicating that S is a circular set of G and hence $cr(G) = a + 2$. As Z is a subset of each cr-set of G, $f_{cr}(G) \leq cr(G) - |Z| = a + 2 - 2 = a$, according to Theorem 2.3. Consequently, $f_{cr}(G) \leq a$. Given that $cr(G) = a = 2$, Furthermore, it is evident that every cr-set of G that contains Z has the form $S = Z \cup \{c_1, c_2, \dots, c_a\}$, where $c_i \in H_i$ $(1 \leq i \leq a)$. Given $|T| < a$, let T be any suitable subset of S. After that, c_j $(1 \leq j \leq a)$ is a vertex such that $c_j \notin T$. Assume that b_j, a vertex of H_j, is different from c_j. Subs equently, $S_1 = (S - \{c_j\}) \cup \{b_j\}$ is a cr-set that correctly contains T. T is not a forced subset of S as a result. For every minimum cr-set of G, this is true. Consequently, $f_{cr}(G) = a$.

Next, we prove that $f_g(G) = a$. The representation of every extreme vertex in G is $Z_1 = Z \cup \{w_1, w_3\}$. Theorem 1.1 states that every g-set in G is a subset of Z_1. Give H_i: $\{r_i, s_i\}$ $(1 \leq i \leq a)$. Since every g-set of G contains at least one vertex from every $H_i (1 \leq i \leq a)$, it is easy to demonstrate that $g(G) \geq a + 4$. $S = Z_1 \cup \{r_1, r_2, \dots, r_a\}$ is assumed. Consequently, since $I[S] = V(G)$, S is a g-set of G and $g(G) = a + 4$. All of the g-sets in G have a subset called Z_1. The $f_g(G) \leq a$ Theorem applies. Z_1 appears in every g-set of G, and since $g(G) = a + 4$, it follows that every g-set of G has the form $S = Z_1 \cup \{c_1, c_2, \dots, c_a\}$, where $c_i \in H_i (1 \leq i \leq a)$. Let T be any appropriate subset of S such that $|T| < a$. After that, a vertex such that is $c_j \notin T$ is c_j $(1 \leq j \leq a)$. Presume that t_j, one of H_j's vertices, is distinct from c_j. Consequently, a g-set that suitably contains T is $S_1 = (S - \{c_j\}) \cup \{b_j\}$. As such, T is not a forced subset of S. This is valid for any smallest g-set of G. As a result, $f_g(G) = a$.

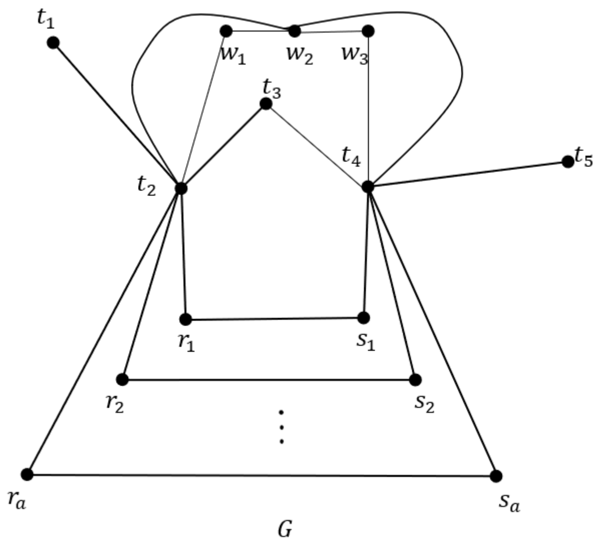

Figure 3.4

Theorem 3.5. Let G be a connected graph. For every integer $a \geq 0$, and $b \geq 0$, there exists $f_g(G) = a$ and $f_{cr}(G) = b$.

Proof. Case (i) $a = 0, b \geq 1$. The graph produced in Theorem 3.3 meets the requisite requirement.

Case (ii) $a \geq 1, b = 0$. The graph constructed Theorem 3.3, satisfies the required condition.

Case (iii) $a = b \geq 1$. The graph constructed Theorem 3.4, satisfies the required condition.

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Case (iv) $0 < a < b$. Consider the graph G given in Figure 3.5.

We first establish that $f_g(G) = a$. The set of all extreme vertex of G is denoted by $Z = \{t, w_1, w_3, x_1, x_2, \dots, x_{b-a}, y_1, y_2, \dots, y_{b-a}\}$. Thus Z is a subset of each geodetic set in G, according to Theorem 3.4. H_i: $\{r_i, s_i\}$ $(1 \le i \le a)$ be given. Every geodetic set of G has exactly one vertex from each H_i $(1 \le i \le a)$, as can be seen easily. As a result, $(G) \ge 3 + b - a + b - a + a = 2b - a + 3$. Let $S = Z \cup \{r_1, r_2, \dots, r_a\}$. Thus, S is a geodetic set of G since $I[S] = V(G)$. Consequently, $g(G) = 2b - a + 3$. $f_g(G) \le g(G) - |Z| = 2b - a + 3 - (2b - 2a + 3) = a$, according to the theorem. Given that $g(G) = 2b - a + 3$ and that Z_1 appears in every g-set of G, it is clear that every g-set of G has the form $S = Z \cup \{c_1, c_2, \dots, c_a\}$where $c_i \in H_i$ $(1 \le i \le a)$. Given $|T| < a$, let T be any suitable subset of S. After that, c_j $(1 \le j \le a)$ is a vertex such that $c_j \notin T$. Assume that b_j, a vertex of H_j, is different from c_j. Subsequently, $S_1 = (S - \{c_j\}) \cup \{b_j\}$ is a g-set that correctly contains T. Such that T is not a forced subset of S as a result. For every smallest g-set of G, this holds true. Hence, $f_g(G) = a$.

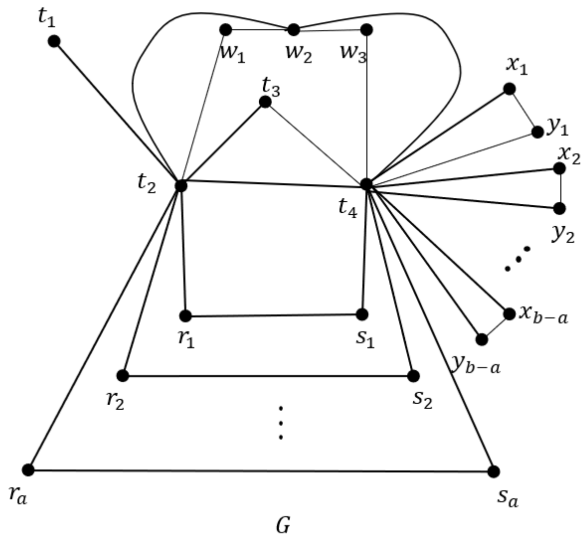

Figure 3.5

We then demonstrate that $f_{cr}(G) = b$. Let $Z_1 = \{t\}$ represent G's end vertex. Such that Z is a subset of every cr-set in G according to Theorem 1.1. Q_j: $\{x_j, y_j\}$ $(1 \le j \le b - a)$ be given. Every circular set of G has exactly one vertex from every Q_j $(1 \le j \le b - a)$ and exactly one vertex from every H_i $(1 \le i \le a)$, as can be seen easily. Therefore, $cr(G) \ge 1 + a + b - a = b + 1$. Assume that $S = Z \cup \{r_1, r_2, \dots, r_a\} \cup \{x_1, x_2, \dots, x_{b-a}\}$. As a result, S is a circular set of G since $I[S] = V(G)$. Consequently, $cr(G) = b + 1$. Therefore $f_{cr}(G) \le cr(G) - |Z| = b + 1 - 1 = b$, according to the theorem. Every circular set of G, of which Z is a subset, has the form $W_1 = Z \cup \{c_1, c_2, \dots, c_a\} \cup \{d_1, d_2, \dots, d_{b-a}\}$, where $d_j \in Q_j$ $(1 \le j \le b - a)$ and $c_i \in H_i$ $(1 \le i \le a)$. Given $|T| < b$, let T be any proper subset of W_1. After that, $c_j, d_j \notin T$ since there are vertices $c_j \in H_i$ and $d_j \in Q_j$. Assume that f_j is a vertex of Q_j that is separate from d_j and that e_i is a vertex of H_i apart from c_i. $W_2 =$

$\left(W_1 - \{c_j, d_j\}\right) \cup \{e_i, f_j\}$ is a cr-set that correctly contains T in this case. Such that T is not a forced subset of W_2 as a result. For every minimum cr-set of G, this is true. Thus, $f_{cr}(G) = b$.

Case (v) $0 < b < a$.

First, we establish that $f_{cr}(G) = b$. Assume $Z = \{t_1, \mathrm{w}\}$. Thus Z is therefore a subset of G's cr-set. Assume $H_i{:}\{r_i, s_i\}$ $(1 \le i \le b)$be given. It is simple to see that $cr(G) \ge b + 2$ as every circular set of G has at least one vertex from each $H_i (1 \le i \le a)$. Consider $S = Z_1 \cup \{r_1, r_2, \dots, r_b\}$. Consequently, S is a circular set of G since $I[S] = V(G)$, and $cr(G) = b + 2$. Given that Z is a subset of each G cr-set. According to Theorem $f_{cr}(G) \le cr(G) - |Z| = +2 - 2 = b$. Thus, $f_{cr}(G) \le b$. It is clear that every cr-set of G is of the type $S = Z_1 \cup \{c_1, c_2, \dots, c_a\}$, where $c_i \in H_i (1 \le i \le a)$, since $cr(G) = b + 2$ and every cr-set of G contains Z. Given $|T| < a$, let T be any proper subset of S. After that, c_j $(1 \le j \le a)$ is a vertex such that $c_j \notin T$. Assume that b_j, a vertex of H_j, is different from c_j. Following that, $S_1 = \left(S - \{c_j\}\right) \cup \{b_j\}$ is a cr-set that correctly contains T. Such that T is not a forced subset of S as a result. For every minimum cr-set of G, this is true. Hence, $f_{cr}(G) = a$.

We then demonstrate that $f_g(G) = a$. Let $Z = \{t_1, t_3, w_1, w_3\}$ represent all of G's extreme vertices. Consider Z is a subset of each geodetic set in G, according to Theorem 3.4. Z should be $Z = Z_1 \cup \{w\}$. Therefore Z_1 is clearly a subset of each and every geodetic set in G. Assume $Q_j{:}\{u_j, v_j\}$ $(1 \le j \le a - b)$. Every geodetic set of G has at least one vertex from each of the H_j $(1 \le j \le a)$ and each of the Q_j, as can be clearly recognised; so, $g(G) \ge 4 + a - b + b = 4 + a$. Assume $W = Z_1 \cup \{r_1, r_2, \dots, r_b, u_1, u_2, \dots, u_{a-b}\}$. Consequently, $g(G) = a + 4$ since $I[W] = V(G)$ and W is a geodetic set of G. $f_g(G) \le g(G) - |Z| = a + 4 - 4 = a$ according to Theorem 1.2. It is evident that every g-set of G is of the form $W_1 = Z \cup \{c_1, c_2, \dots, c_b\} \cup \{d_1, d_2, \dots, d_{a-b}\}$ since Z is a subset of every g-set of G. In this case, $d_j \in Q_j$ $(1 \le j \le a - b)$ and $c_i \in H_i$ $(1 \le i \le b)$. Given $|T| < b$, let T be any proper subset of W_1. After that, $c_j, d_j \notin T$ since there are vertices $c_i \in H_i$ and $d_j \in Q_j$. Assume Q_j is a vertex of H_i that is different from d_j and c_i. $W_2 = \left(W_1 - \{c_j, d_j\}\right) \cup \{e_j, f_j\}$ is a cr-set that correctly contains T in this case. Such that T is not a forced subset of S as a result. For every minimum cr-set of G, this is true. Therefore, $f_g(G) = a$.

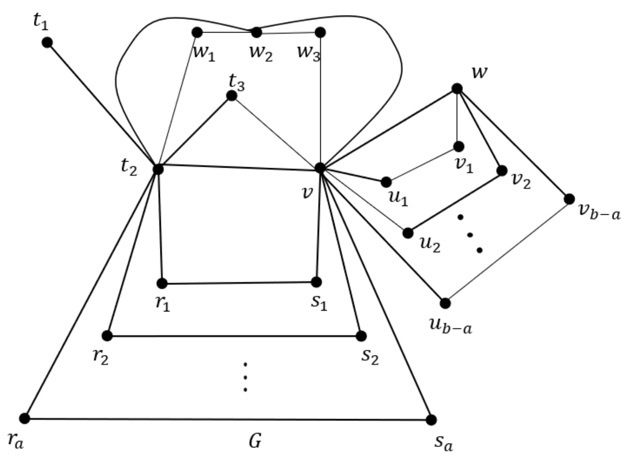

Figure 3.6

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