Analysis of Sierpinski Triangle Based on Fuzzy Triangular Numbers and Dihedral Group

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Abstract Fractals are indefinitely complex patterns such as self-similar across at different scales; for example, Sierpinski triangle is a fractal. This paper analysed in the Sierpinski triangle. It is considered as equilateral triangles such as 1 unit, k unit and $k + 1$ unit. Each iteration is divided as [(0, 1/4, ½, …, 1)], [(0, k/4, k/2…, k)], [(0, ($k + 1$)/4, ($k + 1$)/2,… ($k + 1$))], so on. It analysed this triangle which satisfies fuzzy triangular numbers and the number of the theoretical aspect of fuzzy triangular numbers (FTNs) in self-similarity set of fractal set (Sierpinski triangle) and some arithmetic operations of α-ut and discussed that this triangle satisfied the centroid and median of the normal triangle. Multiplication of fuzzy triangular numbers α-cuts is explained graphically. It also analysed that these smaller equilateral triangles form a group, and this group satisfies the property of dihedral group.

Keywords Fuzzy numbers · Fractal · Sierpinski triangle · Fuzzy triangular numbers · Centroid of the triangle · Dihedral group

AMS Classification 2000 · 03E72 · 28A80 · 20D15

1 Introduction

In earlier times, there was no mathematical concept to describe the uncertain (vagueness) situation. Fuzzy numbers are defined in uncertainty situation, and it is playing a vital role in many applications, but the main problem in the development of the application is the computational complexity. Hence, more attention is needed to simply arithmetic computation with fuzzy numbers. By restricting the fuzzy number to triangular fuzzy numbers, addition and subtraction become simpler [1].

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1.1 Fuzzy Number

Fuzzy number, which is the extension of real numbers, has its properties which can be related to the theory of numbers [2]. It is widely used in engineering applications because of their suitability for representing uncertain information [3].

1.2 Fractals

The word fractal was coined by Mandelbrot; *fractus* means broken, to describe the objects that were too irregular to fit into a traditional geometrical setting [4, 5]. Many fractals have characteristics of self-similarity that it made up parts to resemble the whole in some way. The few examples of fractals are such as Cantor sets, Von Koch curve, Menger sponge, dragon curve, Julia sets and Mandelbrot sets [6].

1.3 Sierpinski Triangle

In 1915, Waclaw Sierpinski described the triangle which is a self-similar structure, and it occurs at different levels of iterations or magnifications; it is named as Sierpinski triangle. The Sierpinski triangle is a fractal, and it satisfies all the properties of a fractal [4]. For the triangular area, with each iteration, the side of the inside triangle reduces by a factor of 2. The numbers of these little triangles, on the other, had increased not by 4 but by a factor of 3. The dimension of self-similar object is then (log 3/log 2) = 1.58 approximately [7].

1.4 Dihedral Group

The dihedral groups play a significant role in the group theory, while the dihedral groups are originally produced from the symmetries of regular polygons, which together from surfaces and planes. A regular polygon which has rotations and reflections forms a *d*ihedral group, and the dihedral group for n-polygon is denoted by $D_{2n.}$ [8, 9].

Fuzzy triangular number as a tool for student assessment is explained in this paper [10]. New operation on triangular fuzzy number for solving fuzzy linear programming problem is discussed [3]. It explained about triangular approximation of fuzzy numbers—a new approach [1], and they all discussed with randomly chosen triangular fuzzy numbers. In Sect. 2, operations of fuzzy triangular numbers with graphical representation are discussed using Sierpinski triangle with side $n = 1$, k and $k + 1$ unit and also discussed dihedral group in fuzzy triangular number triangle.

2 Operations on Fuzzy Triangular Numbers

2.1 Iteration of Equilateral Triangle

Figure 1 represents the iteration of the Sierpinski triangle from stage 0 to 3.

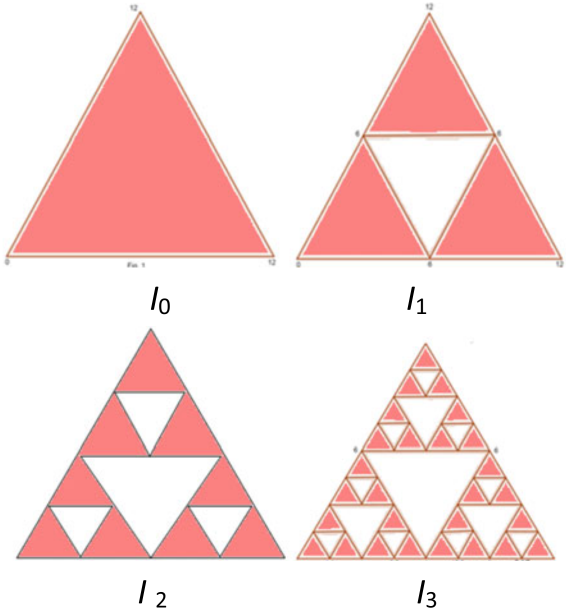

Fig. 1 Iteration of Sierpinski triangle

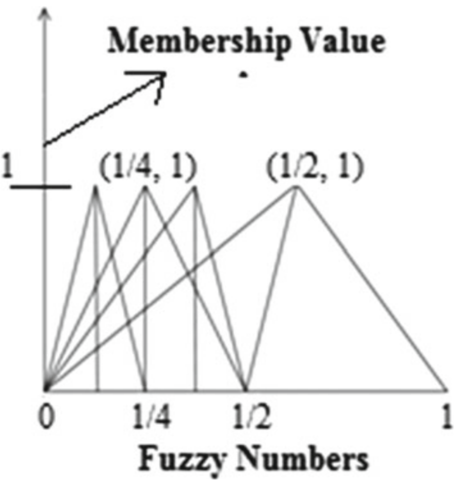

Fig. 2 (1 unit)

Figures 2, 3 and 4 represent fuzzy triangular numbers obtained from an e*quilateral triangle* of side 1 unit, k unit and $k + 1$ unit, respectively.

Based on Fig. 1, the iteration of the Sierpinski triangle of side 1 unit, k units and $k + 1$ unit is explained in Tables 1, 2 and 3, respectively.

Fig. 3 (k unit)

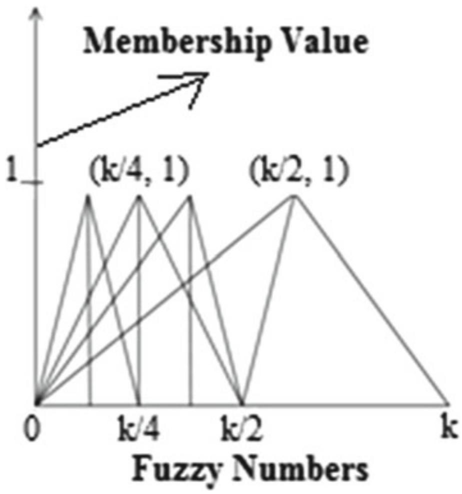

Fig. 4 ($k + 1$ unit)

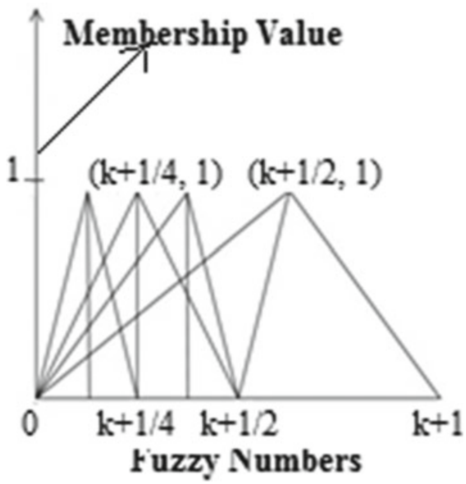

Table 1 Iteration of Sierpinski triangle of side 1 unit

Iteration number	Scaling	TFNs
I_1	$\frac{1}{2}$	$(0, \frac{1}{2}, 1)$
I_2	$\frac{1}{4}$	$(0, \frac{1}{4}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{2}, 1), (0, \frac{1}{8}, \frac{1}{4}),$
I_3	$\frac{1}{8}$	…
I_n	$\left(\frac{1}{2}\right)^n$	n–numbers

Table 2 Iteration of Sierpinski triangle of side K unit

Iteration number	Scaling	TFNs
I_1	$\frac{1}{2}$	$(0, \frac{K}{2}, k)$
I_2	$\frac{1}{4}$	$(0, \frac{k}{4}, \frac{k}{2}), (\frac{k}{4}, \frac{k}{2}, k), (0, \frac{k}{8}, \frac{k}{4}),$
I_3	$\frac{1}{8}$	…
I_n	$\left(\frac{1}{2}\right)^n$	n–numbers

Table 3 Iteration of Sierpinski triangle of side $K + 1$ unit

Iteration number	Scaling	TFNs
I_1	$\frac{1}{2}$	$(0, \frac{k+1}{2}, k + 1)$
I_2	$\frac{1}{4}$	$(0, \frac{k+1}{4}, \frac{k+1}{2}), (\frac{k+1}{4}, \frac{k+1}{2}, k + 1), (0, \frac{k+1}{8}, \frac{k+1}{4}),.$
I_3	$\frac{1}{8}$	…
I_n	$\left(\frac{1}{2}\right)^n$	n–numbers

2.2 *FTN in* α-cut Operations

By using the definition of [11], let *a, b* and *c* be real numbers with $a < b < c$. *T*hen, the TFN $\grave{A} = (a, b, c)$ is a fuzzy number with membership function

$$m(\Phi) = \begin{cases} \frac{\Phi - a}{b - a}, & a \le \Phi \le b \\ \frac{c - \Phi}{c - b}, & b \le \Phi \le c \\ 0, & \Phi < a,\ \Phi > c \end{cases}$$

By using the definition of [10, 12], the α-cuts $\grave{A}_\alpha$ of a TFN $\grave{A} = (a, b, c)$, $\alpha \in [0, 1]$, then

$$\grave{A}_\alpha = \left[\grave{A}_i\alpha,\ \grave{A}_r\alpha\right] = \left[(a + \alpha(b - a)),\ (c - \alpha\,(c - b))\right]$$

α-cut operation and from Fig. 3 and from Table 2, $G\ (0, k/2, k)$

$$G_\alpha = [a_1\alpha,\ a_3\,\alpha] = (k\,\alpha,\ -\frac{k}{2}\alpha + k) \tag{1}$$

Operation by α-cut $F\ (0, \frac{k}{4}, \frac{k}{2})$

$$F_\alpha = [a_1\alpha,\ a_3\,\alpha] = (\frac{k}{4}\alpha,\ -\frac{k}{4}\alpha + \frac{k}{2}) \tag{2}$$

Approximation for multiplication from Eqs. (1) and (2)

$$G_\alpha(\cdot)\,F_\alpha = [(\frac{k}{2}\alpha \cdot \frac{k}{4}\alpha),\ (-\frac{k}{2}\alpha + k \cdot -\frac{k}{4}\alpha + \frac{k}{2})] \tag{3}$$

$$\alpha = 0 = (0,\ k^2/2)$$
$$\alpha = 1 = (k^2/8,\ k^2/8)$$

$$G_\alpha(\cdot)\,F_\alpha = (0,\ k^2/2,\ k^2/8)$$

From Fig. 4 and from Table 3 and based on definition TFN α-cut

$$A\ (0,\ \frac{k+1}{2},\ k+1)$$

$$A_\alpha = [a_1\alpha,\ a_3\alpha] = [\{(k+1)\,\alpha,\ -\frac{k+1}{2}\,\alpha + k + 1\}] \tag{4}$$

Operation by α-cut $B\ (0,\ \frac{k+1}{4},\ \frac{k+1}{2})$

$$B_\alpha = [a_1\alpha,\ a_3\alpha] = [(\frac{k+1}{2})\,\alpha,\ (-\frac{k+1}{4})\alpha + \frac{k+1}{2})] \tag{5}$$

Approximation for multiplication from Eqs. (4) and (5)

$$A_\alpha.B_\alpha = [\{(k+1)\alpha \cdot (\frac{k+1}{2})\alpha\},\ \{(-\frac{k+1}{2}\alpha + k + 1) \cdot (-\frac{k+1}{4}\alpha + \frac{k+1}{2})\}]$$
$$\alpha = 0 = (0,\ \frac{(k+1)2}{2})$$
$$\alpha = 1 = \{\frac{(k+1)2}{2},\ \frac{(k+1)2}{8}\}$$

$$A_\alpha\,(.)\,B_\alpha = [0,\ \frac{(k+1)2}{2},\ \frac{(k+1)2}{8}] \tag{6}$$

Therefore multiplication of two TFN is Triangular Fuzzy Number but it is not always. The lines connecting the end points are parabolic and triangular form in actual product and standard approximation, respectively.

2.3 Graphical Representation of Multiplication of FTN

Figures 5 and 6 represent a graphical representation of side$k$ units and $k + 1$ units, respectively.

Consider two fuzzy numbers $U = (0, k/2, k)$, $\tilde{U} = (0, k/4, k/2)$, and from Eqs. (1), (2) and (3), multiplication of two fuzzy numbers [1] based on this lemma their respective α*-cut will be* $U(\alpha) = \frac{k^2\alpha}{8}$, $\tilde{U}(\alpha) = (k^2\,\alpha^2\text{-}\,4k^2\alpha + 4k^2)\,/\,8$,$v(r) = \frac{k^2}{16} + \frac{k^2 r}{16}$, $v\,{*}(r) = \frac{5k^2}{16} - \frac{3k^2 r}{2}$.

Consider two fuzzy numbers $(0, \frac{k+1}{2}, k + 1)$, $(0, \frac{k+1}{4}, \frac{k+1}{2})$, and from the Eqs. (4), (5) and (6) and the lemma [1], α-cut will be

$$U(\alpha) = [\frac{(k+1)^2\alpha^2]}{8},\ \tilde{U}(\alpha) = \frac{(k+1)^2\alpha^2]}{8} - \frac{(k+1)^2\alpha]}{2} + \frac{(k+1)^2]}{2},$$
$$\nu(r) = \frac{(k+1)^2]}{16} + \frac{(k+1)^2 r]}{4},\ \nu^*(r) = 5\frac{(k+1)^2]}{16} - 3\frac{(k+1)^2 r]}{8}$$

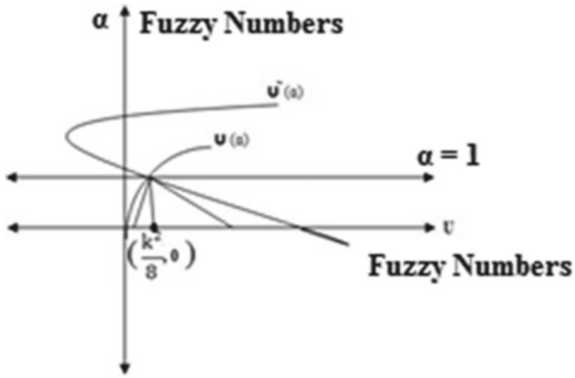

Fig. 5 α-cut diagram for k

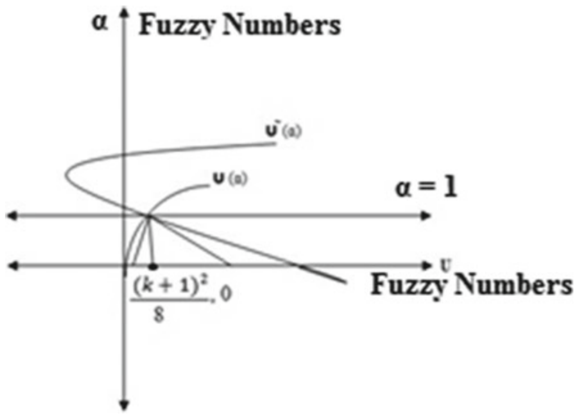

Fig. 6 α-cut diagram of $k + 1$

2.4 *Calculating Centroid and Median of FTN Triangle of Side* k, k + *1 Units*

In Fig. 7, Let ABC is a Fuzzy triangle obtained from FTN of side k unit and G is the centroid of the triangle ABC of side k units with the vertices $A(0{,}0)$, $B(k/2,\ 1)$, and $C(k,\ 0)$ and $M(k/2,\ 0)$ is the midpoint of A and C, and $N(3\ k/4,\ 1/2)$ is the midpoint of B and C. Based on the definition TFN and from Fig. 6, Equation of the straight line $A(0,\ 0)$ and $N((3\ k)/4,\ 1/2)$ on which AN lies in

$$2x - 3ky\ = 0 \tag{7}$$

From Fig. 7, based on the definition TFN, Equation of B (k/2, 1) and M (k/2,0)lies in

$$2x + 0y = k \tag{8}$$

The linear system of Eqs. (7), and (8) has a consistent and has unique solution determining the coordinates of the triangle COG in Fig. 7 and the centroid of the triangle after observing the following

$$D = \begin{vmatrix} 2 & -3k \\ 2 & 0 \end{vmatrix} = 0 + 6k = 6k \tag{9}$$

$$D_x = \begin{vmatrix} 0 & -3k \\ K & 0 \end{vmatrix} = 3k^2 \tag{10}$$

$$Dy = \begin{vmatrix} 2 & 0 \\ 2 & k \end{vmatrix} = 2k \tag{11}$$

Therefore, $x = k/2$, $y = 1/3$. Solving Eqs. (7) and (8), based on the definition [10, 13].

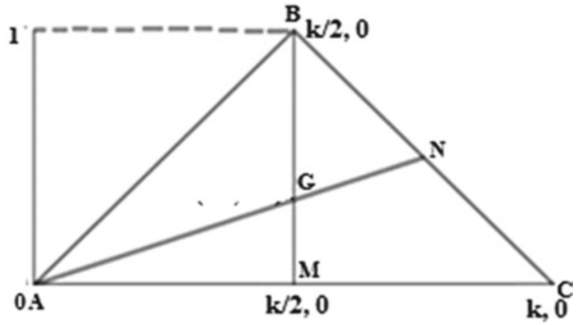

Fig. 7 Fuzzy triangular numbers triangle with centroid and median

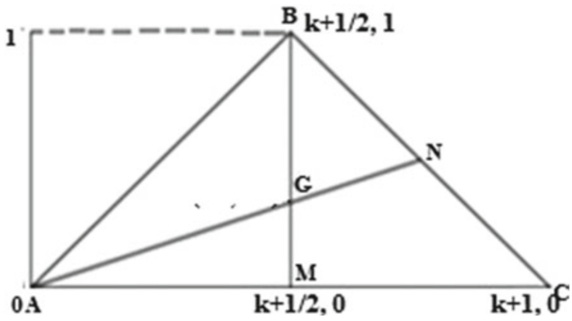

Fig. 8 Fuzzy triangular numbers triangle with centroid and median

[$x = (a + b + c)/3$, $y = 1/3$]. The coordinates of centroid of the triangle G are ($k/2$, 1/3) which is equal to $x = \frac{(0+k+k/2)}{3}$, $y = \frac{(1+0+0)}{3}$, i.e. ($k/2$, 1/3). Similarly, for $k + 1$ units.

In Fig. 8, G represents the centroid of the fuzzy triangular numbers triangle of side, $k + 1$units with the vertices $A(0, 0)$, $B(k + 1/2, 1)$, and $C(k + 1,0)$ and $M(\frac{k+1}{2}$, 0) is the midpoint of A and C, and $N((3k + 3)/4, 1/2)$ is the midpoint of B and C. $x = \frac{(0+\frac{k+1}{2}+(k+1))}{3}$, $y = \frac{(1+0+0)}{3}$, i.e. $(\frac{k+1}{2}$, 1/3).

2.5 Dihedral Group in FTN

From Figs. 2, 3 and 4, the FTNs triangle obtained after the iteration it has six symmetries i.e. three rotations and three reflections; asymmetry can interchange some of the sides and vertices. Figure 9 represents rotations and reflections of the triangle which is obtained from the equilateral triangle after each iteration. (Assume that the sides are 1, 2, 3, respectively) ®, ꓤrepresents reflections, rotations, respectively.

Each rotation has binary composite functions because every image has a unique pre-image. This can be represented by matrix also $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$.

Table 4 clearly shows that it satisfies the group property of **Closure:** If $®_1$ and $®_2$ are in the group, then $®_1 \circ ®_2$are also in the group. **Associativity:** If ꓤ$_1$, ꓤ$_2$ *and* ꓤ$_3$are in the group, then ((ꓤ$_1 \circ$ ꓤ$_2$) $\circ$ ꓤ$_3$ = ꓤ$_1 \circ$ (ꓤ$_2 \circ$ ꓤ$_3$)) **Identity:** There is an element of the group such that a $\circ$ e = e e $\circ$ a = a, for all a belongs the group, here e = $®_0$. **Inverse:** For any element a of the group, there is an element a^{-1} such that a $\circ$ a^{-1} = **e** and a^{-1} $\circ$ a = e, and every element has inverse but not a commutative because ꓤ$_3 \circ ®_2 \neq ®_2 \circ$ ꓤ$_3$(rotation and reflection will not commute in general); therefore, it is a non-abelian group.

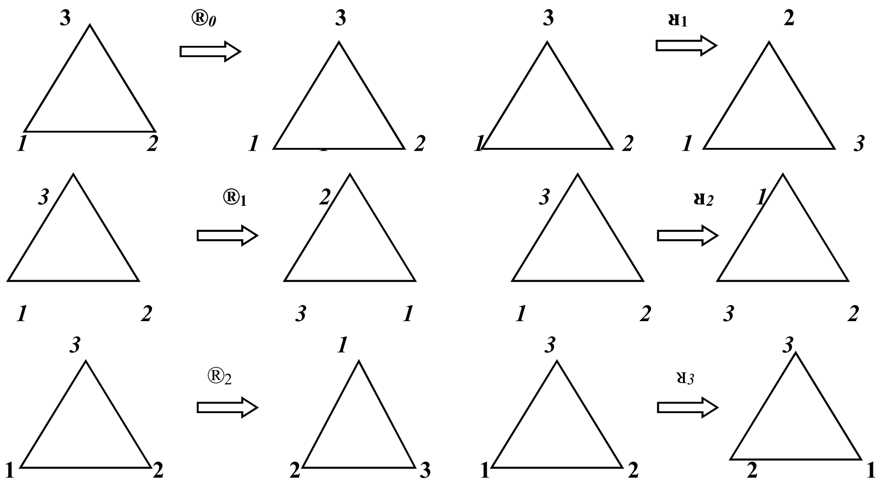

Fig. 9 Rotations and reflections of FTNs triangle

Table 4 Cayley table of usual multiplication

o	®$_0$	®$_1$	®$_2$	ꓤ$_1$	ꓤ$_2$	ꓤ$_3$
®$_0$	®$_0$	®$_1$	®$_2$	ꓤ$_1$	ꓤ$_2$	ꓤ$_3$
®$_1$	®$_1$	®$_2$	®$_0$	ꓤ$_3$	ꓤ$_1$	ꓤ$_2$
®$_2$	®$_2$	®$_0$	®$_1$	ꓤ$_2$	ꓤ$_3$	ꓤ$_1$
ꓤ$_1$	ꓤ$_1$	ꓤ$_2$	ꓤ$_3$	®$_0$	®$_1$	®$_2$
ꓤ$_2$	ꓤ$_2$	ꓤ$_3$	ꓤ$_1$	®$_2$	®$_0$	®$_1$
ꓤ$_3$	ꓤ$_3$	ꓤ$_1$	ꓤ$_2$	®$_1$	®$_2$	®$_0$

3 Conclusion

Multiplication of fuzzy triangular numbers α-cuts is discussed in self-similarity of Sierpinski triangle. This triangle satisfied fractal property. From this, it concludes that fuzzy triangular numbers triangle satisfied FTN properties for all value of n where n belongs to natural number, and multiplication of two fuzzy triangular numbers is approximation of fuzzy triangular numbers. Median and centroid derived from the triangle formed by fuzzy triangular numbers triangle are same as a normal triangle, and it has six symmetries (rotations and reflections); therefore, it forms dihedral group of non-abelian. This dihedral group plays an important role in group theory, chemistry (the group is used in the classification of molecules and crystals) and geometry.

This research article aims is that proposed and bridged some new ideas in mathematics with fuzzy concept and also developing and strengthening the number-theoretical aspects of fuzzy numbers in future.

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