Advances in Mathematics: Scientific Journal **10** (2021), no.2, 891–899
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.2.18

HOMOMORPHISM ON FUZZY MODULAR *L*-IDEAL AND THE ROLE OF FUZZY MODULAR *L*-FILTERS IN COMMUTATIVE *L*-*M* GROUP

D. Vidyadevi [1] and S. Meenakshi

ABSTRACT. This paper explores the concept of homomorphism on fuzzy modular $l-$ideals. We derived the fundamental theorem of homomorphism. This article also introduces the notion of fuzzy modular $l-$filters in commutative $l-m$ group with suitable example and study the characterization theorem for fuzzy modular $l-$filters. In addition some results and properties are also discussed.

1. INTRODUCTION

L. A. Zadeh [5] proffered the most suitable mathematical kit to deal with lack of certainty as Fuzzy sets. Mordeson and Malik [2] developed to fuzzy commutative algebra. A brief foundation to the set of all fuzzy normal subgroups of a group constitute a sub lattice of a lattice of all fuzzy subgroups of a given group and is modular was established by N. Ajmal. It was later independently introduced by Das that fuzzy subgroups by their level subgroups. The fuzzy setting has several applications in various domains. Nanda proposed the notion of fuzzy lattice using the concept of fuzzy partial ordering.

[1] *corresponding author*

2020 *Mathematics Subject Classification.* 06D72, 06C05, 06F15, 03B52, 06B10.

Key words and phrases. fuzzy modular $l-$ideals, cosets, fuzzy modular $l-$filters, level $l-$filters.

Submitted: 08.01.2021; *Accepted:* 06.02.2021; *Published:* 22.02.2021.

G.S.V. SatyaSaibaba [4] initiate the study of $L-$fuzzy lattice ordered groups and sub $l-$groups. J. A. Goguen [1] replaced the valuation set $[0,1]$ by means of a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying $L-$ fuzzy sets. M.U. Makandar and A.D. Lokhande [7] extended the concept to fuzzy lattice ordered m group.

Also, many authors have worked on fuzzy lattice theory. R. Natarajan and J.Vimala [3] introduced the notion of ideals and distributive $l-$ideals and $l-$filters in commutative $l-$ groups. U.M. Swamy and D.ViswanandaRaju introduced the conception of the theory of fuzzy ideals and gave some interesting results. SG.Karpagavalli and D. Vidyadevi [6] has proposed the idea of fuzzy modular l-ideal in commutative l-m group. The idea of partially ordered algebraic systems play vital role in algebra. Lattice ordered groups and lattice ordered rings are some important concepts in partially ordered systems. These concepts play a prominent role in with wide ranging applications in many disciplines.

2. Preliminaries

Definition 2.1. *A non-empty subset I of G is called an modular $l-$ideal of G if,*

(i) I *is a* m *subgroup of* G.
(ii) I *is a modular sub lattice of* G.
(iii) $0 < mx < a$ **and** $a \in I \implies mx \in I$.

Definition 2.2. *A fuzzy set is a pair (X, μ) where X is any non-empty set and defined as $\mu : X \to [0,1]$.*

Definition 2.3. *Let (X, μ) be a fuzzy set and $t \in [0,1]$. Then for any fuzzy m group of G, we define the set $\mu_t = \{mx/x \in G, m \in M, M \subseteq G, \mu(mx) \geq t\}$ is called level set of μ.*

3. Homomorphism on fuzzy modular $l-$ideals

Definition 3.1. *Let G be a commutative $l-m$ group and $\mu : G \to [0,1]$ be the fuzzy modular $l-$ideal. Then the fuzzy set $(mx)\mu : G \to [0,1]$ defined by $((mx).\mu)(my) = \mu((my)(mx)) = \mu(m(yx))$ is called a coset of the fuzzy modular $l-$ideal of μ. The set of all cosets of μ in G is denoted by G_μ.*

Proposition 3.1. *Let G and G' be two commutative $l-m$ groups. $f : G \to G'$ be any function. Let μ_1 be the fuzzy modular $l-$ideal of G and μ_2 be the fuzzy modular $l-$ideal of G'. Define the pre image of f as $f^{-1}[(\mu_2)] = \mu_2[f(mx)]$ for all mx in G. Then $f^{-1}(\mu_2)$ is a fuzzy modular $l-$ideal of G'.*

Proof.

$$
\begin{aligned}
(i) \quad f^{-1}[\mu_2(mx \vee my)] &= \mu_2[f(mx \vee my)] \\
&= \mu_2[f(mx) \vee f(my)] \\
&\geq \mu_2[f(mx)] \wedge \mu_2[f(my)] \\
&= f^{-1}[\mu_2(mx)] \wedge f^{-1}[\mu_2(my)] \\
\implies f^{-1}[\mu_2(mx \vee my)] &= f^{-1}[\mu_2(mx)] \wedge f^{-1}[\mu_2(my)]
\end{aligned}
$$

$$
\begin{aligned}
(ii) \quad f^{-1}[\mu_2(mx \wedge my)] &= \mu_2[f(mx \wedge my)] \\
&= \mu_2[f(mx) \wedge f(my)] \\
&\geq \mu_2[f(mx)] \wedge \mu_2[f(my)] \\
&= f^{-1}[\mu_2(mx)] \wedge f^{-1}[\mu_2(my)] \\
\implies f^{-1}[\mu_2(mx \vee my)] &= f^{-1}[\mu_2(mx)] \wedge f^{-1}[\mu_2(my)]
\end{aligned}
$$

$$
\begin{aligned}
(iii) \quad f^{-1}[\mu_2(mx)(my)] &= \mu_2[f(mx) \wedge (my)] \\
&= \mu_2[f(mx) \wedge f(my)] \\
&\geq \mu_2[f(mx)] \wedge \mu_2[f(my)] \\
&= f^{-1}[\mu_2(mx)] \wedge f^{-1}[\mu_2(my)] \\
\implies f^{-1}[\mu_2(mx)(my)] &= f^{-1}[\mu_2(mx)] \wedge f^{-1}[\mu_2(my)]
\end{aligned}
$$

$$
\begin{aligned}
(iv) \quad & f^{-1}[\mu_2(mx \vee my) \wedge \mu_2(mx \vee mz)] \\
&= \mu_2[f(mx \vee my) \wedge f(mx \vee mz)] \\
&= \mu_2[f(mx) \vee [f(my) \wedge f(mx \vee mz)]] \\
&\geq \mu_2[f(mx)] \vee \mu_2[f(my) \wedge f(mx \vee mz)] \\
&= f^{-1}[\mu_2(mx)] \vee f^{-1}[\mu_2(my) \wedge \mu_2(mx \vee mz)].
\end{aligned}
$$

Let $0 < mx < a$ in G' Since μ_2 is the fuzzy modular$l-$ideal of G', then we have $\mu_2(mx) \geq \mu_2(a)$. Then,

$$\implies f^{-1}[\mu_2(mx)] = \mu_2[f(mx)] \geq \mu_2[f(a)] = f^{-1}[\mu_2(a)]$$

$\implies f^{-1}[\mu_2(mx)] = f^{-1}[\mu_2(a)]$

Hence $f^{-1}(a)$ is a fuzzy modular $l-$ideal of G'. $\square$

Proposition 3.2 (Fundamental theorem of homomorphism)**.** *Let μ be any fuzzy modular $l-$ideal of the commutative $l-m$ group G. Then $G/\mu_t \cong G_\mu$ where $t = \mu(0)$.*

Proof. Let μ be any fuzzy modular $l-$ideal of G and G_μ be the set of all cosets of μ in G. Define $f : G \to G_\mu$ by $f(mx) = (mx)\mu$.

(i) f is well defined.

Let $mx = my$. Then, $(mx)\mu = (my)\mu$ and, further, $f(mx) = f(my)$.

(ii) f is onto.

Let $(mx)\mu \in G_\mu$. Then there exists $mx \in G$ such that $f(mx) = (mx)\mu$.

(iii) f preserves $\vee$, $\wedge$, $\bullet$.

$$\begin{aligned}
f((mx)(my)) &= ((mx)(my))\mu \\
&= ((mx)\mu)((my)\mu) \\
&= f(mx)f(my) \\
f(mx \vee my) &= (mx \vee my)\mu \\
&= ((mx)\mu) \vee ((my)\mu) \\
&= f(mx) \vee f(my) \\
f(mx \wedge my) &= (mx \wedge my)\mu \\
&= ((mx)\mu) \wedge ((my)\mu) \\
&= f(mx) \wedge f(my) \\
f(mx \vee my) &\wedge f(mx \vee mz) \\
&= (mx \vee my)\mu \wedge (mx \vee mz)\mu \\
&= (mx)\mu \vee (my)\mu \wedge (mx \vee mz)\mu \\
&= f(mx) \vee [f(my) \wedge f(mx \vee mz)]
\end{aligned}$$

Suppose $mx \in kerf$ if and only if $\mu(mx) = \mu(0)$. Then,

$\implies \mu(mx) = \mu(0) = t \implies mx \in \mu_t \implies kerf = \mu_t.$

Hence $G/\mu_t \cong G_\mu$. $\square$

4. Fuzzy modular $l-$filters in commutative $l-m$ group

Definition 4.1. *Let G be the commutative $l-m$ group. A non-empty subset F of G is called an modular $l-$filter of G if,*

(i) F *is a* m *subgroup of* G*;*

(ii) F *is a modular sub lattice of* G*;*

(iii) $0 < mx < a$ and $mx \in F \implies a \in F$.

Definition 4.2. *Let G be a commutative $l-m$ group. A fuzzy set $\mu : X \rightarrow [0,1]$ is said to be fuzzy modular $l-$filter of G if,*

(i) $\mu(m(xy)) \geq \mu(mx) \wedge \mu(my)$

(ii) $\mu(mx \vee my) \geq \mu(mx) \wedge \mu(my)$

(iii) $\mu(mx \wedge my) \geq \mu(mx) \wedge \mu(my)$

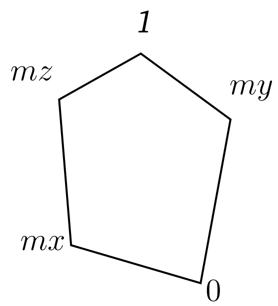
(iv) $\mu(mx \vee my) \wedge \mu(mx \vee mz) \geq \mu(mx) \vee [\mu(my) \wedge \mu(mx \vee mz)]$

(v) $0 < mx < a \implies \mu(a) \geq \mu(mx)$

$\forall\, x, y, z, a, b \in G$, $m \in M$, $M \subseteq G$

Definition 4.3. *Let μ_1 and μ_2 be any two fuzzy modular $l-$filters of G. Then μ_1 is said to be contained in μ_2 denoted by $\mu_1 \subseteq \mu_2$ if $\mu_1(mx) \leq \mu_2(mx)$ for all $mx \in G$. If $\mu_1(mx) = \mu_2(mx)$ for all $mx \in G$ then μ_1 and μ_2 are said to be equal and we can write $\mu_1 = \mu_2$.*

Example 1. *Consider $G = \{0, mx, my, mz, 1\}$. Let $\mu : X \rightarrow [0,1]$ is a fuzzy set in G.*



Let the fuzzy set be $\mu : X \rightarrow [0,1]$ of G, it satisfies all the conditions of fuzzy modular $l-$filter. Then μ is fuzzy modular $l-$filter of G respectively.

Definition 4.4. *Let G be commutative $l-m$ group. For any fuzzy modular $l-$filter of G and let $t \in [0,1]$. Then $\mu_t = \{mx/x \in G, m \in M, M \subseteq G, \mu(mx) \geq t\}$ is called an level modular $l-$filter of μ.*

Proposition 4.1. *Let μ_1 and μ_2 be any two fuzzy modular $l-$filters of G. If $\mu_1 \subseteq \mu_2$ then $\mu_1 \cup \mu_2 = \mu_2$ and $\mu_1 \cap \mu_2 = \mu_1$.*

Proof. Given μ_1 and μ_2 are any two fuzzy modular $l-$filters of G. Assume that $\mu_1 \subseteq \mu_2$. Let $mx \in G$ be arbitrary. Then $\mu_1(mx) \leq \mu_2(mx)$ for all $mx \in G$. We have

$$\mu_1 \cup \mu_2 = \max\{\mu_1(mx), \mu_2(mx)\}\ for\ all\ mx \in G = \mu_2(mx).$$

Then, $\mu_1 \cup \mu_2 = \mu_2$.

Also $\mu_1 \cap \mu_2(mx) = \min\{\mu_1(mx),\, \mu_2(mx)\} = \mu_1(mx) \implies \mu_1 \cap \mu_2 = \mu_1$. □

Proposition 4.2. *Let μ_1 and μ_2 be any two fuzzy modular $l-$filters of G. Then $\mu_1 \cup \mu_2 \supseteq \mu_1 \cap \mu_2$.*

Proof. Let $mx \in G$ be arbitrary. Then

$$\begin{aligned}(\mu_1 \cup \mu_2)(mx) &= \max\{\mu_1(mx), \mu_2(mx)\}\\ &\geq \min\{\mu_1(mx), \mu_2(mx)\}\\ &= (\mu_1 \cap \mu_2)(mx)\\ \implies \mu_1 \cup \mu_2 &\supseteq \mu_1 \cap \mu_2.\end{aligned}$$

□

Definition 4.5. *If μ_1 and μ_2 be any two fuzzy modular $l-$filters of G, the join of μ_1 and μ_2 is defined by $(\mu_1 \vee \mu_2) = \sup_{(mx=my\vee mz)}\{\min\{\mu_1(my), \mu_2(mz)\}\}$, where $mx, my, mz \in G$. The meet of $\mu_1 and \mu_2$ is defined by*

$$(\mu_1 \wedge \mu_2) = \sup_{(mx=my\vee mz)}\{\min\{\mu_1(my), \mu_2(mz)\}\},$$

where $mx, my, mz \in G$.

Proposition 4.3. *If μ_1 and μ_2 are any two fuzzy modular $l-$filters of G, then $\mu^\alpha : G \to [0,1]$ defined by $\mu^\alpha(mx) = (\mu(mx))^\alpha$ is also a fuzzy modular $l-$filter.*

Proof.

$$\begin{aligned}\mu^\alpha(m(xy)) &= (\mu(m(xy)))^\alpha\\ &= (\mu(mx))^\alpha \wedge (\mu(my))^\alpha\\ &= \mu^\alpha(mx) \wedge \mu^\alpha(my)\\ \implies \mu^\alpha(m(xy)) &= \mu^\alpha(mx) \wedge \mu^\alpha(my).\end{aligned}$$

Similarly, $\mu^\alpha(mx \wedge my) = \mu^\alpha(mx) \wedge \mu^\alpha(my)$ and $\mu^\alpha(mx \wedge my) = \mu^\alpha(mx) \wedge \mu^\alpha(my)$. Now, let $mx, my \in G$ such that $mx = my$. Since μ is a fuzzy $l-$filter $\mu(mx) = \mu(my)$, we have that $\mu^\alpha(mx) = (\mu(mx))^\alpha = \mu^\alpha(my) = (\mu(my))^\alpha$, i.e., $\mu^\alpha(mx) = \mu^\alpha(my)$. Hence μ^α is also fuzzy modular $l-$filter. □

Proposition 4.4. *Let $\mu : X \to [0,1]$ be a subset of G. If two level modular $l-$filters μ_t, μ_s for $t, s \in [0,1]$ of fuzzy modular $l-$filter of G are equal with $t < s$ then there is no mx in G such that $t < \mu(mx) < s$.*

Proof. Let μ_t and μ_s level modular $l-$filters. Assume that they are equal such that

$$\mu_t = \mu_s.$$

Let prove there is no mx in G such that $t < \mu(mx) < s$. On the contrary, suppose there exists $mx \in G$ such that $t < \mu(mx) < s$. Then, $\mu_s \subseteq \mu_t$. If $mx \in \mu_t$, then mx may not belong to μ_s which is a contradiction. Therefore there is no $mx \in G$ such that $t < \mu(mx) < s$. Conversely assume that there is no mx in G such that

$$\begin{aligned} &t < \mu(mx) < s, \\ &\mu_t = \{mx \in G/\mu(mx) \leq t\} \text{ and } s \geq t, \\ &\mu_s = \{mx \in G/\mu(mx) \leq s\}. \end{aligned}$$

Then

$$\mu_s \subseteq \mu_t. \tag{4.1}$$

It is enough to show that $\mu_t \subseteq \mu_s$. Let $mx \in \mu_s$. Then $\mu(mx) \leq s$, and further, $\mu(mx) \leq t$, $mx \in \mu_s$, and

$$\mu_t \subseteq \mu_s. \tag{4.2}$$

From (4.1) and (4.2), $\mu_t = \mu_s$. Hence the two level modular $l-$filters are equal. □

Proposition 4.5. *(Characterization theorem for fuzzy modular $l-$filters) Let G be a commutative $l - m$ group. A fuzzy set (G, μ) is a fuzzy modular $l-$filter of G if and only if the set $\mu_t = \{mx/x \in G, m \in M, M \subseteq G, \mu(mx) \geq t\}$ is an modular $l-$filter of G for all $t \in [0,1]$ with $\mu_t \neq \phi$.*

Proof. Let μ be a fuzzy modular $l-$filter of G. Let $\mu_t = \{mx/x \in G, m \in M, M \subseteq G, \mu(mx) \geq t, t \in [0,1]\}$. Let $mx, my \in \mu_t$ be arbitrary. Then,

$$\mu(mx) \geq t \text{ and } \mu(my) \geq t.$$

Now, $\mu(m(xy)) \geq \min\{\mu(mx), \mu(my)\} \geq t$. This implies $\mu(m(xy)) \geq t$, and further, $m(xy) \in \mu_t\mu_t$ is a m subgroup of G.

Let $mx, my, mz \in \mu_t$. Then,

$\implies \mu(mx) > t$ and $\mu(my) > t$ and $\mu(mz) > t$

$\implies \mu(mx) \vee \mu(my) \geq t$ and $\mu(mx) \wedge \mu(my) \geq t$

$\implies \mu(mx \wedge my) \geq \min\{\mu(mx, my)\} \geq t$

$\implies mx \wedge my \in \mu_t$.

Also $\mu(mx \vee my) \geq \min\{\mu(mx, my)\} \geq t$, implies $mx \vee my \in \mu_t$. And $\mu(mx \vee my) \wedge \mu(mx \vee mz) \geq \min\{\mu(mx), \mu(my) \wedge \mu(mx \vee mz)\}$,

$\implies (mx \vee my) \wedge (mx \vee mz) \in \mu_t$

$\implies \mu_t$ is a modular sublattice.

Let $a \in \mu_t$. Assume that $0 < mx < a$. Then $mx \in \mu_t$ implies $\mu(mx) \geq t$. Since $mx < a$, $\mu(mx) \leq \mu(a)$,

$\implies \mu(a) \geq t \implies a \in \mu_t$.

Hence μ_t is modular $l-$filter of G.

Conversely, assume that μ_t is modular $l-$filter of G. Let $\min\{\mu(a), \mu(b)\} = r$. Either $\mu(a) = r, \mu(b) \geq \mu(a)$ (or) $\mu(b) = r, \mu(a) \geq \mu(b)$. $\implies \mu(a) \geq r$ and $\mu(b) \geq r$

$\implies a, b \in \mu_r$

$\implies a \bullet b, a \vee b, a \wedge b, (a \vee b) \wedge (a \vee b) \in \mu_r$

$\implies \mu(ab) \geq r, \mu(a \wedge b) \geq r, \mu(a \vee b) \geq r$ and $\mu(a \vee b) \wedge \mu(a \vee b) \geq r$.

Let $a, mx \in G$ and $0 < mx < a$. Let $\mu(mx) = t$.

$\implies mx \in \mu_t \implies a \in \mu_t \implies \mu(a) \geq t = \mu(mx) \implies \mu(mx) \leq \mu(a)$

Hence μ_t is a fuzzy modular $l-$filter of G. □

Proposition 4.6. *If μ is a fuzzy modular $l-$filter of G and $t, s \in [0,1]$, then $\mu_t = \mu_s$, if, and only if, $t = s$.*

Proof. If $t = s$, then clearly $\mu_t = \mu_s$. Conversely, let $\mu_t = \mu_s$. Since $t \in [0,1]$, there exists some $mx \in G$ such that $\mu(mx) = t$.

$\implies mx \in \mu_t \implies mx \in \mu_t = \mu_s \implies mx \in \mu_s$

$\implies \mu(mx) = s \implies t = s.$

Similarly $s = t$. Hence $t = s$. □

5. Conclusion

In this manuscript, we explored the conception of homomorphism on fuzzy modular $l-$ideals and investigated fundamental homomorphism theorem for fuzzy modular$l-$ideals in G. We also proposed the notion of fuzzy modular $l-$filters in G. Also the characterization theorem is discussed and shown some related properties.

References

[1] J. A. Goguen: *L-Fuzzy Sets*, Journal of Math. Anal and Appl., (1967), 145–174.

[2] J. N. Mordeson, D. S. Malik: *Fuzzy Commutative Algebra*, World Scientific Publishing, 1998.

[3] R. Natarajan, J. Vimala: *Distributive l-ideal in Commutative Lattice Ordered Group*, Acta Ciencia Indica Mathematics, **33**(2) (2007), 517-516.

[4] G. S. V. Sathya Saibaba: *Fuzzy Lattice Ordered Groups*, Southeast Asian. Math., **32**(4) (2008), 749-766.

[5] L. A. Zadeh: Fuzzy Sets, Information and Control, **8** (1965), 69–78.

[6] SG. Karpagavalli, D. Vidyadevi: *Fuzzy modular l-ideals in commutative $l-m$ group*, Journal of critical reviews, **7**(7) (2020), 750-752.

[7] M. U. Makandar, A. D. Lokhande: *Fuzzy lattice ordered $m-$ group*, Jorunal of computer application, **82**(8) (2013), 31-36.

Department of Mathematics
Vels Institute of Science, Technology and Advanced Studies
Chennai, Tamilnadu, India
Email address: vidyadayalan04@gmail.com

Department of Mathematics
Vels Institute of Science, Technology and Advanced Studies
Chennai, Tamilnadu, India
Email address: meenakshikarthikeyan8@gmail.com