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Compatible Mappings of Type (γ) and (δ)

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Abstract— In this article, we introduce the definition of two different types of compatible mappings and prove common fixed point theorems in fuzzy metric spaces. Examples are given to support the results proved herein.

Index Terms— fuzzy metric space; compatible mapping; common fixed point.

I. INTRODUCTION

The authors defined intuitionistic (ψ,η) contractive mapping in [7]. Using the definition of ψ, we gave a common fixed point theorem. The generalization of the commuting mapping concept is compatible mapping which is introduced by Gerald Jungck[4]. This concept was generalized to fuzzy metric spaces by Mishra et al.[8]. Y. J. Cho introduced the concept of compatible mapping of type (α)[1] and compatible mapping of type (β)[2]. In this article, we introduce compatible mapping of type (γ) and compatible mapping of type (δ). Further, the theorem is discussed for two different types of compatible mappings. In our paper[7], ψ is defined as follows,

Let Ψ be the class of all mappings $\psi : [0,1] \to [0,1]$ such that

(i) ψ is non-decreasing and $\lim_{n\to\infty} \psi^n(s) = 1, \forall s \in (0,1]$;

(ii) $\psi(s) > s$, $\forall s \in (0,1)$;

(iii) $\psi(1) = 1$;

Example 1.1. [7] Define $\psi : [0,1] \to [0,1]$ by

$$\psi(s) = \frac{2s}{s+1}, \forall s \in [0,1].$$

$$\psi^2(s) = \frac{4s}{3s+1}, \psi^3(s) = \frac{8s}{7s+1}, \ldots,$$

$$\psi^n(s) = \frac{2^n s}{(2^n-1)s+1}, \forall s \in [0,1].$$

$$\lim_{n\to\infty} \psi^n(s) = \lim_{n\to\infty} \frac{2^n s}{(2^n-1)s+1} = 1, \forall s \in (0,1].$$

Clearly, $\psi(s) > s$, $\forall s \in (0,1)$ and $\psi(1) = 1$.

II. PRELIMINARIES

Definition 2.1.[6] Let X be an arbitrary set, ∗ be a continuous t-norm, and M be fuzzy sets on $X^2\times(0,\infty)$. Consider the following conditions ∀ u,v,w ∈ X and t > 0,

(i) $M(u,v,0) = 0$;

(ii) $M(u,v,t) = 1$ if and only if $u = v$;

(iii) $M(u,v,t) = M(v,u,t)$;

(iv) $M(u,w,t + s) \geq M(u,v,t) * M(v,w,s)$;

(v) $M(u,v,t) : (0,\infty) \to [0,1]$ is left continuous;

The pair (M,∗) is called fuzzy metric on X. The triple (X,M,∗) is called a fuzzy metric space.

Example 2.2. [3] Let (X,d) be a metric space. Denote $a*b = ab$, $\forall a,b \in [0,1]$ and let M_d be fuzzy set on $X \times X \times (0,+\infty)$ defined as follows:

$M_d(u,v,t) = \dfrac{t}{t+d(u,v)}, \forall t > 0$, then $(X,M_d,*)$ is a fuzzy metric space.

Definition 2.3. [5] Let (X,M,∗) be a fuzzy metric space. A sequence $\{u_n\}$ in X is called

(a) convergent to a point u ∈ X if and only if

$\lim_{n\to\infty} M(u_n,u,t) = 1, \forall t > 0$,

(b) Cauchy if $\lim_{n\to\infty} M(u_n,u_{n+p},t) = 1, \forall t > 0$ and $p > 0$.

Definition 2.4. [5] A fuzzy metric space is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.5. [8] In a fuzzy metric space (X,M,∗), two self mappings A and B are said to be compatible if $\lim_{n\to\infty} M(ABu_n,BAu_n,t) = 1$ whenever u_n is a sequence in X such that $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = w$ for some w ∈ X.

Definition 2.6. [9] Two self maps A and B of a fuzzy metric space (X,M,∗) are called reciprocally continuous on X if

$\lim_{n\to\infty} ABu_n = Aw$ and $\lim_{n\to\infty} BAu_n = Bw$ whenever $\{u_n\}$ is a sequence in X such that $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = w$ for some w in X.

Lemma 2.7. [8] If A and B are compatible mappings on a fuzzy metric space X and $Au_n, Bu_n \to w$ for some w in X (u_n being a sequence in X) then $ABu_n \to Bw$ provided B is continuous (at w).

III. MAIN RESULTS

Definition 3.1. Let (X,M,∗) be a fuzzy metric space. We say that The two self mappings A and B are called

(a) compatible of type (γ) if for all t > 0, $\lim_{n\to\infty} M(AAu_n,Bw,t) = 1$ and $\lim_{n\to\infty} M(BBu_n,Aw,t) = 1$ whenever $\{u_n\}$ is a sequence in X such that $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = w$ for some w ∈ X.

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(b) compatible of type (δ) if for all t > 0, $\lim_{n\to\infty} AAu_n = \lim_{n\to\infty} ABu_n = Bw$ and $\lim_{n\to\infty} BBu_n = \lim_{n\to\infty} BAu_n = Aw$, whenever $\{u_n\}$ is a sequence in X such that $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = w$ for some w ∈ X.

Example 3.2. Let X = [0,∞) with the metric d defined by d(u,v) = |u − v|, define $M(u,v,t) = \frac{t}{t+d(u,v)}$, ∀ u,v ∈ X and t > 0. Define the self maps A and B by Au = 2u − 1 and $Bu = u^2$. Let $u_n = 1 - \frac{1}{n}$.

$$\lim_{n\to\infty} Au_n = \lim_{n\to\infty} 2u_n - 1 = \lim_{n\to\infty} 2(1 - \frac{1}{n}) - 1 = 1.$$

$$\lim_{n\to\infty} Bu_n = \lim_{n\to\infty} u_n^2 = \lim_{n\to\infty} (1 - \frac{1}{n})^2 = 1.$$

Also,

$\lim_{n\to\infty} AAu_n = \lim_{n\to\infty} 4u_n - 3 = \lim_{n\to\infty} 4(1 - \frac{1}{n}) - 3 = 1 = B1 = Bw$. $\lim_{n\to\infty} BBu_n = \lim_{n\to\infty} u_n^4 = \lim_{n\to\infty} (1 - \frac{1}{n})^4 = 1 = A1 = Aw$. $\lim_{n\to\infty} ABu_n = \lim_{n\to\infty} 2u_n^2 - 1 = \lim_{n\to\infty} 2(1 - \frac{1}{n})^2 - 1 = 1 = B1 = Bw$.

$\lim_{n\to\infty} BAu_n = \lim_{n\to\infty} (2u_n - 1)^2 = \lim_{n\to\infty} 2(1 - \frac{1}{n}) - 1)^2 = 1 = A1 = Aw$.

Therefore, A and B are compatible mapping of type (γ) and also compatible mapping of type (δ).

Proposition 3.3. Let A and B be compatible mapping of type (δ) of a fuzzy metric space (X,M,∗) into itself. Let one of A and B be continuous. Suppose that $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = w$ for some w ∈ X. Then Aw = Bw.

Proof. Let $\{u_n\}$ be a sequence in X such that $\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = w$ for some w ∈ X. Since A and B are compatible mapping of type (λ) , we have $\lim_{n\to\infty} AAu_n = \lim_{n\to\infty} ABu_n = Bw$.

If A is continuous, we get $\lim_{n\to\infty} AAu_n = Aw = \lim_{n\to\infty} BBu_n = \lim_{n\to\infty} BAu_n = Bw$. That is Aw = Bw.

Theorem 3.4. Let A,B,S and T be self-mappings of a complete fuzzy metric space (X,M,∗) with a ∗ b = min(a,b) satisfy the conditions:

(I) BX ⊂ SX, AX ⊂ TX,

(II) M(Au,Bv,t)≥ψ[min{M(Su,Tv,t),M(Au,Su,t),M(Bv,Tv,t), M(Su,Bv,2t),M(Au,Tv,t)}], for all u,v ∈ X and t > 0.

Suppose that A,S and B,T are reciprocally continuous and compatible of type (γ), then A,B,S and T have a unique common fixed point.

Proof. Consider a point u_0 ∈ X. Since BX ⊂ SX and AX ⊂ TX, We can define a sequence $\{v_n\}$ in X as follows:

there exists u_1 ∈ X such that $Au_0 = Tu_1 = v_0$.

there exists u_2 ∈ X such that $Bu_1 = Su_2 = v_1$.

there exists u_{2n+1} ∈ X such that $Au_{2n} = Tu_{2n+1} = v_{2n}$.

there exists u_{2n+2} ∈ X such that $Bu_{2n+1} = Su_{2n+2} = v_{2n+1}$.

Now, for all t > 0,

$$\begin{aligned} M(v_{2n},v_{2n+1},t) &= M(Au_{2n},Bu_{2n+1}) \\ &\geq \psi[\min\{M(Su_{2n},Tu_{2n+1},t),M(Au_{2n},Su_{2n},t), M(Bu_{2n+1},Tu_{2n+1},t),M(Su_{2n},Bu_{2n+1},2t), M(Au_{2n},Tu_{2n+1},t)\}] \\ &= \psi[\min\{M(v_{2n-1},v_{2n},t), M(v_{2n},v_{2n-1},t), M(v_{2n+1},v_{2n},t), M(v_{2n-1},v_{2n+1},2t), M(v_{2n},v_{2n},t)\}] \\ &= \psi[\min\{M(v_{2n-1},v_{2n},t), M(v_{2n+1},v_{2n},t), [M(v_{2n-1},v_{2n},t)* M(v_{2n},v_{2n+1},t)],1\}] \\ &= \psi[M(v_{2n-1},v_{2n},t),M(v_{2n},v_{2n+1},t)] \\ &= \psi[\min\{M(v_{2n-1},v_{2n},t),M(v_{2n},v_{2n+1},t)\}]. \end{aligned}$$

since ψ(s) > s, for all s ∈ (0,1),

$M(v_{2n},v_{2n+1},t) \geq \psi(M(v_{2n},v_{2n+1},t)) > M(v_{2n},v_{2n+1},t)$

which is a contradiction.

Therefore, $M(v_{2n},v_{2n+1},t) \geq \psi(M(v_{2n-1},v_{2n},t))$.

That is, $M(v_n,v_{n+1},t) \geq \psi(M(v_{n-1},v_n,t))$.

Continuing this process, we can get

$M(v_n,v_{n+1},t) \geq \psi(M(v_{n-1},v_n,t)) \geq \psi^2(M(v_{n-2},v_{n-1},t)) \geq \cdots \geq \psi^n(M(v_1,v_0,t))$.

That is, $M(v_n,v_{n+1},t) \geq \psi^n(M(v_1,v_0,t))$.

Taking limit as n → ∞ and $\lim_{n\to\infty} \psi^n(s) = 1$, for all s ∈ (0,1], $\lim_{n\to\infty} M(v_n,v_{n+1},t) = 1$.

Similarly, we can prove $M(v_{n+1},v_{n+2},t) \geq \psi^n(M(v_2,v_1,t))$.

$\lim_{n\to\infty} M(v_{n+1},v_{n+2},t) = 1$.

Now for all p > 0,

$$M(v_n,v_{n+p},t) \geq M(v_n,v_{n+1},\frac{t}{p}) * \ldots * M(v_{n+p-1},v_{n+p},\frac{t}{p}).$$

Taking limit n → ∞, we have,

$$\begin{aligned} \lim_{n\to\infty} M(v_n,v_{n+p},t) &\geq \lim_{n\to\infty} M(v_n,v_{n+1},\frac{t}{p}) * \ldots * \lim_{n\to\infty} M(v_{n+p-1},v_{n+p},\frac{t}{p}) \\ &\geq 1 * \ldots * 1 \\ &= 1. \end{aligned}$$

That is, $\lim_{n\to\infty} M(v_n,v_{n+p},t) = 1$.

Hence,$\{v_n\}$ is a Cauchy sequence in X and converges to w and $\lim_{n\to\infty} M(v_n,w,t) = 1$, for each t > 0.

$$\lim_{n\to\infty} Au_{2n} = \lim_{n\to\infty} Tu_{2n+1} = \lim_{n\to\infty} Bu_{2n+1} = \lim_{n\to\infty} Su_{2n+2} = w.$$

Since A,S and B,T are compatible of type (γ), we have $AAu_{2n} \to Sw$, $SSu_{2n} \to Aw$ and $BBu_{2n} \to Tw$, $TTu_{2n} \to Bw$ as $n \to \infty$.

Also Since A and S are reciprocally continuous, $ASu_{2n} \to Aw$, $SAu_{2n} \to Sw$ and since B and T are reciprocally continuous, $BTu_{2n} \to Bw$ and $TBu_{2n} \to Tw$.

We claim that Bw = Aw. Putting $u = Su_{2n}$ and $v = Tu_{2n+1}$ in (II), we have

$$M(ASu_{2n},BTu_{2n+1},t) \geq \psi[\min\{M(SSu_{2n},TTu_{2n+1},t), M(ASu_{2n},SSu_{2n},t), M(BTu_{2n+1},TTu_{2n+1},t), M(SSu_{2n},BTu_{2n+1},2t), M(ASu_{2n},TTu_{2n}+1,t)\}]$$

Taking limit as $n \to \infty$,

$$M(Aw,Bw,t) \geq \psi[\min\{M(Aw,Bw,t), M(Aw,Aw,t), M(Bw,Bw,t), M(Aw,Bw,2t), M(Aw,Bw,t)\}] = \psi[\min\{M(Aw,Bw,t),1\}] = \psi(M(Aw,Bw,t)).$$

That is, $M(Aw,Bw,t) \geq \psi(M(Aw,Bw,t))$.

Since $\psi(s) \geq s$ for all $s \in (0,1]$, it is possible only when $M(Aw,Bw,t) = 1$.

That is, Aw = Bw.

Now, we claim that Sw = Bw. Putting u = w and $v = Tu_{2n+1}$ in (II), we have

$$M(AAw,BTu_{2n+1},t) \geq \psi[\min\{M(SAw,TTu_{2n+1},t), M(AAw,SAw,t), M(BTu_{2n+1},TTu_{2n+1},t), M(SAw,BTu_{2n+1},2t), M(AAw,TTu_{2n+1},t)\}]$$

Taking limit as $n \to \infty$,

$$M(Sw,Bw,t) \geq \psi[\min\{M(Sw,Bw,t), M(Bw,Sw,t), M(Bw,Bw,t), M(Sw,Bw,2t), M(Bw,Bw,t)\}] = \psi[M(Sw,Bw,t)].$$

That is, $M(Sw,Bw,t) \geq \psi(M(Sw,Bw,t))$.

Since $\psi(s) \geq s$ for all $s \in (0,1]$, it is possible only when $M(Sw,Bw,t) = 1$.

That is, Sw = Bw.

We claim that Sw = w. Putting $u = u_{2n}$ and v = w in (II), we have,

$$M(Au_{2n},BTw,t) \geq \psi[\min\{M(Su_{2n},TTw,t), M(Au_{2n},Su_{2n},t), M(Bw,TTw,t), M(Su_{2n},BTw,t), M(Au_{2n},TTw,t)\}]$$

Taking limit as $n \to \infty$, we get

$$M(w,Bw,t) \geq \psi[\min\{M(w,Bw,t), M(w,w,t), M(Bw,Bw,t), M(w,Bw,t), M(w,Bw,t)\}] = \psi[\min\{M(w,Bw,t),1\}] = \psi(M(w,Bw,t)).$$

That is, $M(w,Bw,t) \geq \psi(M(w,Bw,t))$.

Since $\psi(s) \geq s$, for all $s \in (0,1]$, it is possible only when $M(w,Bw,t) = 1$.

That is, w = Bw.

Hence, w = Aw = Bw = Sw = Tw. Therefore, w is a common fixed point of A,B,S and T.

Uniqueness:

Suppose w_1 is also a common fixed point of A,B,S and T.

$$M(w,w_1,t) = M(Aw,Bw_1,t) \geq \psi[\min\{M(Sw,Tw_1,t), M(Aw,Sw,t), M(Bw_1,Tw_1,t), M(Sw,Bw_1,2t), M(Aw,Tw_1,t)\}] = \psi[\min\{M(w,w_1,t), M(w,w,t), M(w_1,w_1,t), M(w,w_1,2t), M(w,w_1,t)\}] = \psi[M(w,w_1,t)].$$

That is, $M(w,w_1,t) \geq \psi[M(w,w_1,t)]$

This is possible only when $M(w,w_1,t) = 1$. That is, $w = w_1$. Hence the proof.

Theorem 3.5. Let A,B,S and T be self-mappings of a complete fuzzy metric space $(X,M,*)$ with $a * b = \min(a,b)$ satisfy the conditions (I) and (II). Suppose that one of A and S is continuous, and one of B and T is continuous.

Assume that A,S and B,T and compatible of type (δ), then A,B,S and T have a unique common fixed point.

Proof. From the previous Theorem, $\{v_n\}$ is a Cauchy sequence in X and converges to w and $\lim_{n\to\infty} M(v_n,w,t) = 1$, for each $t > 0$.

$$\lim_{n\to\infty} Au_{2n} = \lim_{n\to\infty} Tu_{2n+1} = \lim_{n\to\infty} Bu_{2n+1} = \lim_{n\to\infty} Su_{2n+2} = w.$$

Suppose A and S are compatible of type (δ) and one of mappings A and S is continuous, by Proposition 3.3, we have Aw = Sw.

$$M(Aw,Bu_{2n+}1,t) \geq \psi[\min\{M(Sw,Tu_{2n+1},t), M(Aw,Sw,t), M(Bu_{2n+1},Tu_{2n+1},t), M(Sw,Bu_{2n+1},2t), M(Aw,Tu_{2n+1},t)\}].$$

Taking limit as $n \to \infty$, we get

$$M(Aw,w,t) \geq \psi[\min\{M(Aw,w,t),1,M(w,w,t),M(Aw,w,2t), M(Aw,w,t)\}] = \psi[M(Aw,w,t)].$$

That is, $M(Aw,w,t) \geq \psi(M(Aw,w,t))$.

Since $\psi(s) \geq s$, for all $s \in (0,1]$, it is possible only when $M(Aw,w,t) = 1$.

That is, Aw = w.

Thus, Aw = Sw = w.

Suppose B and T are compatible of type (δ) and one of mappings B and T is continuous, by Proposition 3.3, we have Bw = Tw. Now,

$$M(Au_{2n},Bw,t) \geq \psi[\min\{M(Su_{2n},Tu_{2n},t), M(Au_{2n},Su_{2n},t), M(Bw,Tw,t), M(Su_{2n},Bw,2t), M(Au_{2n},Tw,t)\}].$$

Taking limit as $n \to \infty$, we get

$$M(w,Bw,t) \geq \psi[\min\{M(w,w,t), M(w,w,t),1, M(w,Bw,2t), M(w,Bw,t)\}] = \psi[M(w,Bw,t)].$$

That is, $M(w,Bw,t) \geq \psi(M(w,Bw,t))$.

Since $\psi(s) \geq s$, for all $s \in (0,1]$, it is possible only when $M(w,Bw,t) = 1$.

That is, w = Bw.

Thus Bw = Tw = w.

Hence, Aw = Bw = Sw = Tw = w. w is a common fixed point of A,B,S and T.

Uniqueness follows as in previous Theorem.

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