Proyecciones (Antofagasta. On line) | vol. 38, n. 4 (2019) | pp. 699-705

REVISTAS CIENTÍFICAS de la Universidad Católica del Norte. revistas.ucn.cl

doi:10.22199/issn.0717-6279-2019-04-0045

PROYECCIONES Journal of Mathematics

ISSN 0717-6279 (On line)

Equitable total chromatic number of splitting graph

G. Jayaraman* orcid.org/0000-0003-49095269

D. Muthuramakrishnan**

K. Manikandan***

*Vels Institute of Science, Technology and Advanced Studies, Dept. of Mathematics, Chennai, TN, India.
jayaram07maths@gmail.com
**National College (Autonomous), Trichy, TN, India
dmuthuramakrishnan@gmail.com
***Guru Nanak College (Autonomous), Chennai, TN, India
kmanimaths1987@gmail.com

Received: June 2018 | Accepted: March 2019

Abstract:

Among the varius coloring of graphs, the concept of equitable total coloring of graph G is the coloring of all its vertices and edges in which the number of elements in any two color classes differ by atmost one. The minimum number of colors required is called its equitable total chromatic number. In this paper, we determine an equitable total chromatic number of splitting graph of P_n, C_n and $K_{1,n}$.

Keywords: Equitable total coloring; Equitable total chromatic number; Splitting graph.

MSC (2010): 05C15.

Cite this article as (IEEE citation style):

J. Govindasamy, D. Muthuramakrishnan, and K. Manikandan, "Equitable total chromatic number of splitting graph", *Proyecciones (Antofagasta, On line)*, vol. 38, no. 4, pp. 699-705, Oct. 2019, doi: 10.22199/issn.0717-6279-2019-04-0045. [Accessed dd-mm-yyyy].

1. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. In 1994, the concept of total coloring $\chi''(G)$ was introduced by Behzad [1] and Vizing [11]. A total coloring of a graph G is an assignment of colors to both the vertices and edges of G, such that no two adjacent or incident vertices and edges of G are received the same colors. They both conjectured that for any graph G the following inequality holds: $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$, where $\Delta(G)$ is the maximum degree of G. It is clear that $\Delta(G) + 1$ is the possible lower bound. In 1994, Fu [5] first introduced the concept of equitable total coloring and the equitable total chromatic number of a graph. Gong Kun et.al [3] proved some results on the equitable total chromatic number of $W_n \vee K_n$, $F_m \vee K_n$ and $S_m \vee K_n$. In 2012, Ma Gang and ma Ming [6] proved some results concerning the equitable total chromatic number of $P_m \vee S_n$, $P_m \vee F_n$ and $P_m \vee W_n$. Tong et.al [9] proved that the equitable total chromatic number of $C_m \square C_n$. Girija et.al [2] proved that equitable total chromatic number of Double star graph and fan graph. Gang et.al [7] proved that on the equitable total coloring of multiple join graph. Zhang Zhong-fu [13] proved that on the equitable total coloring of some join graphs. Veninstine vivik et.al [10] proved an algorithmic approach to equitable total chromatic number of wheel graph, Gear graph, Helm graph and sunlet graph.

2. Preliminaries

Definition 2.1. *The splitting graph[8] of a graph G is obtained from adding to each vertex v, a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G that is $N(v) = N(v')$. It is denoted by $S'(G)$.*

Definition 2.2. *For a simple graph $G(V, E)$, let f be a proper $k-$ total coloring of G.*

$$||T_i| - |T_j|| \leq 1, i, j = 1, 2, \dots k.$$

The partition $\{T_i\} = \{V_i \cup E_i : 1 \leq i \leq k\}$ is called a $k-$ equitable total coloring and $\chi_{et}(G) = min\, \{k/\, k\text{-equitable total coloring of } G\}$ *is called the equitable total chromatic number of G, where for all $x \in T_i = V_i \cup E_i$, $f(x) = i$, $i = 1, 2, \dots k$.*

Conjecture 2.3([5]) For any simple graph $G(V, E)$,

$$\chi_{et}(G) \leq \Delta(G) + 2.$$

Conjecture 2.4([14]) For any simple graph $G(V, E)$,

$$\chi_{et}(G) \geq \chi''(G) \geq \Delta(G) + 1.$$

Conjecture 2.5([12]) For every graph G, G has an equitable total $k-$ coloring for each $k \geq max\{\chi''(G), \Delta(G) + 2\}$.

Lemma 2.6([4]) For $n \geq 13$, the equitable total chromatic number of Hypo-Mycielski Graph, $\chi_{et}(HM(W_n)) = n + 2$.

In this paper, we determine an equitable total chromatic number of splitting graph of P_n, C_n and $K_{1,n}$.

3. Main Results

Theorem 3.1. *For any positive integer* $n \geq 3$, $\chi_{et}(S'(P_n)) = 5$.

Proof. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n) = \{e_i : 1 \leq i \leq n-1\}$, where $\{e_i : 1 \leq i \leq n-1\}$ be the edges $v_i v_{i+1} (1 \leq i \leq n-1)$. By the definition of splitting graph, introduce the new vertices $\{v_i' : 1 \leq i \leq n\}$ corresponding to the vertices $\{v_i : 1 \leq i \leq n\}$ of P_n, which are added to obtain splitting graph of path $S'(P_n)$. In $S'(P_n)$, the vertex set and the edge set is given by $V(S'(P_n)) = \{v_i : 1 \leq i \leq n\} \cup \{v_i^{'} : 1 \leq i \leq n\}$ and $E(S'(P_n)) = \{e_i : 1 \leq i \leq n-1\} \cup \{e_i^{'} : 1 \leq i \leq n-1\} \cup \{e_i^{''} : 1 \leq i \leq n-1\}$, where $e_i (1 \leq i \leq n-1)$ is an edge $v_i v_{i+1} (1 \leq i \leq n-1)$, $e_i' (1 \leq i \leq n-1)$ is an edge $v_i v_{i+1}^{'} (1 \leq i \leq n-1)$ and $e_i^{''} (1 \leq i \leq n-1)$ is an edge $v_i^{'} v_{i+1} (1 \leq i \leq n-1)$. Now we partition the vertex set and edge set of $S'(P_n)$ as follows. We consider the following two cases,

Case(i): When n is odd

$$T_1 = \{v_1, v_3, \ldots v_n\} \cup \{e_2^{''}, e_4^{''}, \ldots e_{n-1}^{''}\}$$

$$T_2 = \{v_2, v_4, \ldots v_{n-1}\} \cup \{e_1^{''}, e_3^{''}, \ldots e_{n-2}^{''}\}$$

$$T_3 = \{e_1, e_3, \ldots e_{n-2}\} \cup \{v_i^{'} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\}$$

$$T_4 = \{e_2, e_4, \ldots e_{n-1}\} \cup \{v_i^{'} : \left\lceil \frac{n}{2} \right\rceil \leq i \leq n-1\}$$

$$T_5 = \{e_i^{'} : 1 \leq i \leq n-1\} \cup \{v_n{'}\}$$

Clearly T_1, T_2, T_3, T_4 and T_5 are independent sets of $S'(P_n)$. Its satisfies the inequality $||T_i| - |T_j|| \leq 1$ for $i \neq j$. Therefore the graph $S'(P_n)$ is equitably total colored with 5 colors. This implies that $\chi_{et}(S'(P_n)) \leq 5$. Further, since $\Delta = 4$, we have $\chi_{et}(S'(P_n)) \geq \chi''((S'(P_n)) \geq \Delta + 1 \geq 4 + 1 \geq 5$. Hence $\chi_{et}(S'(P_n)) = 5$.

Case(ii): When n is even

$$T_1 = \{v_1, v_3, \dots v_{n-1}\} \cup \{e_2^{''}, e_4^{''}, \dots e_{n-2}^{''}\}$$

$$T_2 = \{v_2, v_4, \dots v_n\} \cup \{e_1^{''}, e_3^{''}, \dots e_{n-1}^{''}\}$$

$$T_3 = \{e_1, e_3, \dots e_{n-1}\} \cup \{v_i^{'} : 1 \leq i \leq \frac{n}{2}\}$$

$$T_4 = \{e_2, e_4, \dots e_{n-2}\} \cup \{v_i^{'} : \frac{n}{2} + 1 \leq i \leq n\}$$

$$T_5 = \{e_i^{'} : 1 \leq i \leq n-1\}$$

Clearly T_1, T_2, T_3, T_4 and T_5 are independent sets of $S'(P_n)$. Its satisfies the inequality $||T_i| - |T_j|| \leq 1$ for $i \neq j$. Therefore the graph $S'(P_n)$ is equitably total colored with 5 colors. This implies that $\chi_{et}(S'(P_n)) \leq 5$. Further, since $\Delta = 4$, we have $\chi_{et}(S'(P_n)) \geq \chi''(S'(P_n)) \geq \Delta + 1 \geq 4 + 1 \geq 5$. Hence $\chi_{et}(S'(P_n)) = 5$. □

Theorem 3.2. *For any positive integer* $n \geq 4$, $\chi_{et}(S'(C_n)) = 5$.

Proof. Let $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $E(C_n) = \{e_i : 1 \leq i \leq n-1\} \cup \{e_n\}$, where $\{e_i : 1 \leq i \leq n-1\}$ be the edges $v_i v_{i+1} (1 \leq i \leq n-1)$ and e_n is an edge $v_n v_1$. By the definition of splitting graph, introduce the new vertices $\{v_i{'} : 1 \leq i \leq n\}$ corresponding to the vertices $\{v_i : 1 \leq i \leq n\}$ of C_n, which are added to obtain splitting graph of cycle $S'(C_n)$. In $S'(C_n)$, the vertex set and the edge set is given by
$V(S'(C_n)) = \{v_i : 1 \leq i \leq n\} \cup \{v_i^{'} : 1 \leq i \leq n\}$ and
$E(S'(C_n)) = \{e_i : 1 \leq i \leq n-1\} \cup \{e_i^{'} : 1 \leq i \leq n-1\} \cup \{e_i^{''} : 1 \leq i \leq n-1\} \cup \{e_n\} \cup \{e_n^{'}\} \cup \{e_n^{''}\}$, where $e_i (1 \leq i \leq n-1)$ is an edge $v_i v_{i+1} (1 \leq i \leq n-1)$, e_n is an edge $v_n v_1$, $e_i{'} (1 \leq i \leq n-1)$ is an edge $v_i v_{i+1}^{'} (1 \leq i \leq n-1)$, $e_n^{'}$ is an edge $v_n v_1^{'}$, $e_i^{''} (1 \leq i \leq n-1)$ is an edge $v_i^{'} v_{i+1} (1 \leq i \leq n-1)$ and $e_n^{''}$ is an edge $v_n^{'} v_1$. Now we partition the vertex set and edge set of $S'(C_n)$ as follows. we consider the following two cases,

Case(i): When n is odd

$$T_1 = \{v_1, v_3, \dots v_{n-2}\} \cup \{e_1^{''}, e_3^{''}, \dots e_{n-2}^{''}\} \cup \{e_{n-1}^{''}\}$$

$$T_2 = \{v_2, v_4, \dots v_{n-1}\} \cup \{e_2'', e_4'', \dots e_{n-3}''\} \cup \{e_n'\} \cup \{e_n''\}$$

$$T_3 = \{e_1, e_3, \dots e_{n-2}\} \cup \{v_i' : 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1\} \cup \{v_n\}$$

$$T_4 = \{e_2, e_4, \dots e_{n-1}\} \cup \{v_i' : \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n\} \cup \{e_1'\}$$

$$T_5 = \{e_i' : 2 \le i \le n-1\} \cup \{e_n\} \cup \{v_1'\}$$

Clearly T_1, T_2, T_3, T_4 and T_5 are independent sets of $S'(C_n)$. Its satisfies the inequality $||T_i| - |T_j|| \le 1$ for $i \ne j$. Therefore the graph $S'(C_n)$ is equitably total colored with 5 colors. This implies that $\chi_{et}(S'(C_n)) \le 5$. Further, since $\Delta = 4$, we have $\chi_{et}(S'(C_n)) \ge \chi''((S'(C_n)) \ge \Delta + 1 \ge 4 + 1 \ge 5$. Hence $\chi_{et}(S'(C_n)) = 5$.

Case(ii): When n is even

$$T_1 = \{v_1, v_3, \dots v_{n-1}\} \cup \{e_1'', e_3'', \dots e_{n-1}''\}$$

$$T_2 = \{v_2, v_4, \dots v_n\} \cup \{e_2'', e_4'', \dots e_n''\}$$

$$T_3 = \{e_1, e_3, \dots e_{n-1}\} \cup \{v_i' : 1 \le i \le \frac{n}{2}\}$$

$$T_4 = \{e_2, e_4, \dots e_n\} \cup \{v_i' : \frac{n}{2} + 1 \le i \le n\}$$

$$T_5 = \{e_i' : 1 \le i \le n\}$$

Clearly T_1, T_2, T_3, T_4 and T_5 are independent sets of $S'(C_n)$. Its satisfies the inequality $||T_i| - |T_j|| \le 1$ for $i \ne j$. Therefore the graph $S'(C_n)$ is equitably total colored with 5 colors. This implies that $\chi_{et}(S'(C_n)) \le 5$. Further, since $\Delta = 4$, we have $\chi_{et}(S'(C_n)) \ge \chi''((S'(C_n)) \ge \Delta + 1 \ge 4 + 1 \ge 5$. Hence $\chi_{et}(S'(C_n)) = 5$. □

Theorem 3.3. *For any positive integer n, $\chi_{et}(S'(K_{1,n})) = 2n + 1,\ n \ge 2$.*

Proof. Let $V(K_{1,n}) = \{v\} \cup \{v_i : 1 \le i \le n\}$, where $\{v_i : 1 \le i \le n\}$ be the pendent vertices and $\{v\}$ be the root vertex of $K_{1,n}$ and $E(K_{1,n}) = \{e_i : 1 \le i \le n\}$, where e_i is an edge $vv_i (1 \le i \le n)$. Now construct the splitting graph of star, introduce the new vertices $\{v'\}$ and $\{v_i' : 1 \le i \le n\}$ corresponding to the vertices $\{v\}$ and $\{v_i : 1 \le i \le n\}$ of $K_{1,n}$, which are added to obtain splitting graph of star $S'(K_{1,n})$. In $S'(K_{1,n})$, the vertex set and the edge sets are given by $V(S'(K_{1,n})) = \{v_i : 1 \le i \le n\} \cup \{v_i{}' : 1 \le i \le n\} \cup \{v\} \cup \{v'\}$ and $E(S'(K_{1,n})) = \{e_i : 1 \le i \le n\} \cup \{e_i' : 1 \le$

$i \leq n\} \cup \{e_i^{''} : 1 \leq i \leq n\}$, where $e_i (1 \leq i \leq n)$ is an edge $vv_i (1 \leq i \leq n)$, $e_i^{'} (1 \leq i \leq n)$ is an edge $vv_i^{'} (1 \leq i \leq n)$ and $e_i^{''} (1 \leq i \leq n)$ is an edge $v'v_i (1 \leq i \leq n)$. Now we partition the vertex set and edge set of $S'(K_{1,n})$ as follows.

$$T_1 = \{v, v'\}$$

$$T_2 = \{e_1^{'}, e_1^{''}, v_n\}$$

$$T_3 = \{e_2^{'}, e_2^{''}, v_1^{'}\}$$

$$T_4 = \{e_3^{'}, e_3^{''}, v_2^{'}\}$$

$$T_5 = \{e_4^{'}, e_4^{''}, v_3^{'}\}$$

....

....

$$T_{n-1} = \{e_{n-2}^{'}, e_{n-2}^{''}, v_{n-3}^{'}\}$$

$$T_n = \{e_{n-1}^{'}, e_{n-1}^{''}, v_{n-2}^{'}\}$$

$$T_{n+1} = \{e_n^{'}, e_n^{''}, v_{n-1}^{'}\}$$

$$T_{n+2} = \{e_1, v_n^{'}\}$$

$$T_{n+3} = \{e_2, v_1\}$$

$$T_{n+4} = \{e_3, v_2\}$$

....

....

$$T_{2n-1} = \{e_{n-2}, v_{n-3}\}$$

$$T_{2n} = \{e_{n-1}, v_{n-2}\}$$

$$T_{2n+1} = \{e_n, v_{n-1}\}$$

Clearly $T_1, T_2, T_3, T_4 \ldots T_{2n+1}$ are independent sets of $S'(K_{1,n})$. Also $|T_2| = |T_3| = |T_4| = ... = |T_{n+1}| = 3$ and $|T_1| = |T_{n+2}| = |T_{n+3}| = \ldots = |T_{2n}| = |T_{2n+1}| = 2$. Its satisfies the inequality $||T_i| - |T_j|| \leq 1$ for $i \neq j$. Therefore the graph $S'(K_{1,n})$ is equitably total colored with 2n+1 colors. This implies that $\chi_{et}(S'(K_{1,n})) \leq 2n + 1$. Further, since $\Delta = 2n$, we have $\chi_{et}(S'(K_{1,n})) \geq \chi''((S'(K_{1,n})) \geq \Delta + 1 \geq 2n + 1$. Hence $\chi_{et}(S'(K_{1,n})) = 2n + 1$. □

References

[1] M. Behzad, *Graphs and their chromatic numbers*, Ph. D. thesis, Michigan State University, East Lansing, MI, USA, 1965.

[2] G. Girija, and J. Veninstine Vivik, "Equitable total coloring of some graphs", *International Journal of Mathematical Combinatorics*, vol. 1, pp. 107-112, Mar. 2015. [On line]. Available: https://bit.ly/338wzRo

[3] G. Kun, Z. Zhongfu and W. Jian Fang, "Equitable total coloring of some join graphs", *Journal of mathematical research and exposition*, vol. 28, no. 4, pp. 823-828, 2008, doi: 10.3770/j.issn:1000-341X.2008.04.010

[4] H. Wang and J. Wei, "The equitable total chromatic number of the graph $HM(W_n)$", *Journal of applied mathematics and computing*, vol. 24, no. 1-2, pp. 313-323, May 2007, doi: 10.1007/BF02832320.

[5] H. Fu, "Some results on equalized total coloring", *Congressus numerantium*, vol. 102, pp. 111-119, 1994.

[6] M. Gang and M. Ming, "The equitable total chromatic number of the some join-graphs", *Open journal of applied sciences*, vol. 2 no. 4B, pp. 96-99, Jan. 2012, doi: 10.4236/ojapps.2012.24B023.

[7] M. Gang, Z. Zhong-fu, "On the equitable total coloring of multiple join graph", *Journal of mathematical research and exposition*, no. 2, pp. 351-354, 2007. [On line]. Available: https://bit.ly/2q0KcE5

[8] E. Sampathkumar and H. Walikar, "On spliting graph of a graph", *Journal of the Karnatak University-Science*, vol. 25-26, pp. 13-16, 1980-1981. [On line]. Available: https://bit.ly/2LYpb5a

[9] T. Chunling, L. Xiaohui, Y. Yuansheng and L. Lizhihe, "Equitable total coloring of $Cm_2 C_n$", *Discrete applied mathematics*, vol. 157, no. 4, pp. 596-601, Feb. 2009, doi: 10.1016/j.dam.2008.08.030.

[10] V. Vivik J. y G. Girija, "An algorithmic approach to equitable total chromatic number of graphs", *Proyecciones (Antofagasta, En línea)*, vol. 36, no. 2, pp. 307-324, Jun. 2017, doi: 10.4067/S0716-09172017000200307

[11] V. Vizing, "On an estimate of the chromatic class of a p− graph", (in Russian), *Diskret. Analiz.*, no. 5, pp. 25-30, 1964.

[12] W. Wang, "Equitable Total Coloring of Graphs with Maximum Degree 3", *Graphs and combinatorics*, vol. 18, no. 3, pp. 677–685, Oct. 2002, doi: 10.1007/s003730200051.

[13] Z. Zhang, W. Wang, S. Bau and J. Li, "On the equitable total coloring of some join graphs", *Journal of information and computational science*, vol. 2, no. 4, pp. 829-834, 2005.

[14] Z. Zhang, J. Zhang, J. Wang, "The total chromatic number of some graph", Scientia Sinica. Series A. 31, 12, pp. 1434-1441, (1988).