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The Generalization of Rook Number r_2 for the Fractal Chessboard

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The Generalization of Rook Number r_2 for the Fractal Chessboard

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Abstract. In this paper we develop a generalized formula of r_2, the number of ways of placing two non-attacking Rooks for the Fractal Chessboard which is defined as a board that grows progressively in a consistent manner using a 2×2 chessboard to its sides and corners. The board is disintegrated into small sub boards based on their position in the whole Fractal Chessboard (FC). The board is disintegrated into sub boards based on their position in the whole board FC. By finding the value of r_2 for each of these sub boards and adding them, the r_2 value of the whole board FC is obtained. Finally the r_2 value is generalized the Fractal Chessboard at any iteration $I \geq 4$.

1. Introduction

Rook theory is the study of permutations with restricted positions. The restriction is described using the terminology from the game of chess. In chess, a Rook can attack any row or column of the 8×8 chess board [2]. Rook theory focuses on the placement of non-attacking rooks in a more general situation [1]. All the possible quadratic polynomials that are the Rook Polynomial for a specific Special Board in two dimension and three dimension has been classified in the recent years [3, 5]. This paper focuses on finding the number of ways of placing two non-attacking Rooks (r_2) in a Fractal Chessboard. As the Fractal [4] Chessboard grows at every iteration by adding the copies of base board $B(2 \times 2)$ in its sides and corner, it expands consistently with every iteration and thus the number of 2×2 boards, rows and columns also increases consistently at each iteration. Due to consistent expansion the number of rows and columns is equal at every iteration for our board. Since the growth of the board is a recursive process, the number of boards in each iteration I can be obtained from the previous iteration.

At iteration 0, we have only the base board B. So $k_0 = 0$ (i.e.) number of copies of B at iteration 0 is zero. The number of boards at each iteration is given by the recursive formula, $k_I = k_{I-1} + 4I$ where k_I is the number of copies of $B(2 \times 2$ base board) at iteration I.

For first iteration $I = 1$, $k_1 = k_0 + 4(1) = 0 + 4 = 4$.

For second iteration $I = 2$, $k_2 = k_1 + 4(2) = 4 + 8 = 12$.

Similarly for other iterations the number of boards can be found using the above recursive formula.

Due to the continuous growth of the board, the r_2 value is obtained by disintegrating the board into small boards for our convenience. Finding the r_2 value of these small boards and adding their values will give the r_2 value of the Fractal Chessboard (FC).
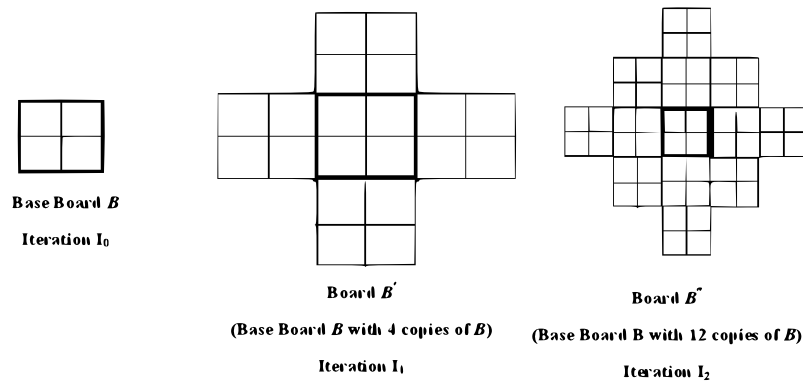
1.1. Notations

I	Iteration number
n_I	Number of 2×2 boards
k_I	Number of copies of base board at I^{th} iteration
r_I	Number of rows
c_I	Number of columns
B	Base board
O	Outer board
A	Board adjacent board to B
A	Board adjacent board to A
O	Board adjacent to O
O	Board adjacent to O
C	Boards lying in the base board B column
C'	Boards adjacent to board C
C''	Boards adjacent to board C' and so on.
R	Remaining boards

Here, the number of rows is equal to the number of columns at each iteration (i.e.) $r = c$.
The boards obtained at each iteration is denoted by B, B', B'', B''' and so on.

The board is categorized as follows:

a) **Base board B:** There is one base board in a fractal chess board from which the other board grows.

b) **Board O:** The board which intersects exactly two rows or two columns are called outer boards and it is denoted by O. There are four outer boards that satisfy the above property in any iteration.

c) **Board A:** The boards which exactly shares one side with the base board B are the boards that are adjacent to B and it is denoted by A. There are four boards in a fractal chess board that shares one of its sides with B in any iteration except at iterations 0 and 1.

d) **Board O':** The boards that are adjacent to the board O are denoted by O'. There are four O' boards in any iteration except at iterations 0, 1 and 2.

e) **Board O'':** The boards that are adjacent to the board O' are denoted by O''. There are eight O'' boards in any iteration except at iterations 0, 1 and 2.

f) **Board A':** The boards that are adjacent to the board A are denoted by A'. There are four A' boards in any iteration except at iterations 0 and 1.

g) **Board C:** The boards which are lying in the base board B column are denoted by C. Number of C boards increases as the iteration increases.

h) **Board C', C'', C'''' and so on:** The boards that are adjacent to C are denoted by C', boards that are adjacent to board C' are denoted by C'' and so on. Number of $C', C'', \dots$ boards increases as the iteration increases.

i) **Remaining boards:** The remaining boards are from the boards that are adjacent to A, A' to the boards that are adjacent to O', O''.

1.2. Fractal Chessboard

Base Board B

Iteration I_0

Board B'

(Base Board B with 4 copies of B)

Iteration I_1

Board B''

(Base Board B with 12 copies of B)

Iteration I_2

The above board grows from the base board B by adding a copy of the base board B to each of its side. If two boards intersect at a point then they share a common board at their point of intersection. The board thus obtained by the above process is called as Fractal Chessboard. Here the board with darker edges is the base board B.

1.3. Disintegration of the Fractal Chessboard (FC)

The Fractal Chessboard (FC) is disintegrated into small boards and these boards are categorized by their positions in FC. The r_2 value is obtained for each of these categorized board. The count of each categorized boards increases at every iteration except for the boards B, O, O', O'', A, A'. The count of these boards are constant at every iteration and their values are 1,4,4,8,4,and 4 respectively.

2. Generalized Formula of r_2 for the Fractal Chessboard

Number of ways of placing one non-attacking Rook in the Fractal Chessboard at any iteration is $4(k_I + 1)$, where k_I is the number of copies of base board B at I^{th} iteration. Number of ways of placing two non-attacking Rooks in the Fractal Chessboard is obtained as follows. The fractal Chessboard FC is divided into small boards. Finding the r_2 value for each divided board and by adding all the r_2 values gives the r_2 value for the Fractal Chessboard FC.

2.1. Value of r_2 for the board B

Placing a Rook in one of the four cells of the base board B, the other non-attacking Rook can be placed in

$$2 + 4\left\{2\left(\frac{r}{2} - 1\right) + 2\left(\frac{r}{2} - 1\right) + 4\left(k - \frac{r}{2} + 1 - \frac{r}{2} + 1\right)\right\}$$
$$= 8(2k - r) + 18 \tag{1}$$

The base board B intersects with $\left(\frac{r}{2} - 1\right)$ boards (row wise) and $\left(\frac{r}{2} - 1\right)$ boards (column wise) of FC.

So each of the $\left(\frac{r}{2} - 1\right)$ boards (rows & columns) comprises two cells to place the second non-attacking Rook.

Thus the sum is $2\left(\frac{r}{2} - 1\right) + 2\left(\frac{r}{2} - 1\right)$.

The remaining $k - \left(\frac{r}{2} - 1\right) - \left(\frac{r}{2} - 1\right)$ boards comprises four cells each. Thus the value adds up to,

$$2\left(\frac{r}{2} - 1\right) + 2\left(\frac{r}{2} - 1\right) + 4\left(k - \left(\frac{r}{2} - 1\right) - \left(\frac{r}{2} - 1\right)\right)$$

The above value is the same for all the four cells of board B.

Therefore, the total value is

$$4\left\{2\left(\frac{r}{2}-1\right)+2\left(\frac{r}{2}-1\right)+4\left(k-\left(\frac{r}{2}-1\right)-\left(\frac{r}{2}-1\right)\right)\right\}$$

Also the board B comprises two cells. Thus the total number of ways of placing the second Rook with the first Rook placed in any of the four cells of board B is given by,

$$2+4\left\{2\left(\frac{r}{2}-1\right)+2\left(\frac{r}{2}-1\right)+4\left(k-\left(\frac{r}{2}-1\right)-\left(\frac{r}{2}-1\right)\right)\right\}$$

It simplifies to,

$$2+4\{4k-2r+4\}$$
$$=18+8(2k-r)$$

2.2. Value of r_2 for the board O

Placing a rook in one of the four cells of the board O, the other non attacking rook can be placed in $4\left[2+4\left\{2\left(\frac{r}{2}-1\right)+4\left(k-\frac{r}{2}+1\right)\right\}\right]$ ways

By simplifying, it gives

$$16(4k-r)+40 \tag{2}$$

The board O intersects with $\left(\frac{r}{2}-1\right)$ boards (row wise) and with no boards (column wise) of W. So each of the $\left(\frac{r}{2}-1\right)$ boards (rows & columns) comprises two cells to place the second non-attacking Rook. Thus the sum is $2\left(\frac{r}{2}-1\right)$.

The remaining $k-\left(\frac{r}{2}-1\right)$ boards comprises four cells each.

Thus the value adds up to,

$$2\left(\frac{r}{2}-1\right)+4\left(k-\left(\frac{r}{2}-1\right)\right)$$

The above value is the same for all the four cells of board O.

Therefore, the sum of the value is

$$4\left\{2\left(\frac{r}{2}-1\right)+4\left(k-\left(\frac{r}{2}-1\right)\right)\right\}$$

Also the board O comprises two cells and there are four outer boards O.

Thus the total number of ways of placing the second Rook with the first Rook placed in any of the four cells of board O is given by,

$$4\left[2+4\left\{2\left(\frac{r}{2}-1\right)+4\left(k-\left(\frac{r}{2}-1\right)\right)\right\}\right]$$

It simplifies to,

$$4\{16k-4r+10\}$$
$$=40+16(4k-r)$$

2.3. Value of r_2 for the board A

Placing a Rook in one of the four cells of the board A, the other non-attacking Rook can be placed in

$$4\left[2 + 4\left\{2 + \left(\frac{r}{2} - 1\right) + 2\left(\frac{r}{3} - 3\right)4\left(k - \frac{r}{2} + 1 - \frac{r}{2} + 3\right)\right\}\right]$$

It Simplifies to

$$32(2k - r) + 136 \tag{3}$$

The board A intersects with $\left(\frac{r}{2} - 1\right)$ boards (row wise) and with $\left(\frac{r}{2} - 3\right)$ boards (column wise) of W. So each of the $\left(\frac{r}{2} - 1\right)$ boards (row wise) and $\left(\frac{r}{2} - 3\right)$ boards (column wise) comprises two cells to place the second non-attacking Rook.

Thus the sum is $2\left(\frac{r}{2} - 1\right) + 2\left(\frac{r}{2} - 3\right)$.

The remaining $k - \left(\frac{r}{2} - 1\right) - 2\left(\frac{r}{2} - 3\right)$ boards comprise four cells each.

By summing up,

$$2\left(\frac{r}{2} - 1\right) + 2\left(\frac{r}{2} - 3\right) + 4\left(k - \left(\frac{r}{2} - 1\right) - \left(\frac{r}{2} - 3\right)\right)$$

This is same for all the four cells of board A.

Therefore, the sum value is

$$4\left\{2\left(\frac{r}{2} - 1\right) + 2\left(\frac{r}{2} - 3\right) + 4\left(k - \left(\frac{r}{2} - 1\right) - \left(\frac{r}{2} - 3\right)\right)\right\}$$

Also the board A comprises two cells and there are four boards adjacent to base board.

Thus the total number of ways of placing the second Rook with the first Rook placed in any of the four cells of board A is,

$$4\left[2 + 4\left\{2\left(\frac{r}{2} - 1\right) + 2\left(\frac{r}{2} - 3\right) + 4\left(k - \left(\frac{r}{2} - 1\right) - \left(\frac{r}{2} - 3\right)\right)\right\}\right]$$

and it simplifies to, $136 + 32(2k - r)$

2.4. Value of r_2 for the board A', O', O''

Similarly,

Value of r_2 for the board A' as

$$32(2k - r) + 200 \tag{4}$$

Value of r_2 for the board O' is given by

$$16(2k - r) - 24 \tag{5}$$

Value of r_2 for the board O'' is given by

$$24(4k - r) + 60 \tag{6}$$

2.5. Value of r_2 for the boards lying in the base board column (C)

$$2\left[2 + 4\left\{2\left(\frac{r}{2} - 1\right) + 2(2l) + 4\left(k - \frac{r}{2} + 1 - 2l\right)\right\}\right] \text{ where } 1 \le l \le \frac{\left(\frac{r}{2} - 7\right)}{2} + 1 \tag{7}$$

2.6. Value of r_2 for the boards that are adjacent to $C, C', C'', \ldots$ so on

$$4 \left[2 + 4 \left\{ 2 \left(\frac{r}{2} - (2m+1) \right) + 2(2l) + 4 \left(k - \left(\frac{r}{2} - (2m+1) \right) - (2l) \right) \right\} \right]$$

$$\text{where } 1 \le l \le \frac{\left(\frac{r}{2} - 7 \right)}{2} + 1, \; 1 \le m \le l \qquad (8)$$

2.7. Value of r_2 for the boards that are adjacent to A, A' to the boards adjacent to O, O'

$$2 \left(2 + 4 \left\{ 2 \left(\frac{r}{2} - 1 \right) + 2(2l) + 4 \left(k - \left(\frac{r}{2} - 1 \right) - (2l) \right) \right\} \right) +$$
$$4 \left(2 + 4 \left\{ 2 \left(\frac{r}{2} - 3 \right) + 2(2l) + 4 \left(k - \left(\frac{r}{2} - 3 \right) - 2l \right) \right\} \right)$$

$$\text{where } 1 \le l \le \frac{\left(\frac{r}{2} - 7 \right)}{2} + 1$$

It reduces to

$$\sum_l 2(48k - 12r + 62 - 48l), \; 1 \le l \le \frac{\left(\frac{r}{2} - 7 \right)}{2} + 1 \qquad (9)$$

2.8. Value of r_2 for the board at Iteration 0, 1, 2 and 3
For iteration 0 the value of r_2 for the base board B which is given by the equation (1),

$$8(2k - r) + 18$$

For first iteration 1, adding the equations (1) and (2), the value of r_2 is

$$8(10k - 3r) + 58$$

For second iteration 2, adding the equations (1), (2), (3) and (4), the value of r_2 is

$$8(26k - 11r) + 394$$

For third iteration 2, adding the equations (1), (2), (3), (4), (5) and (6), the value of r_2 is

$$16(21k - 8r) + 430 \qquad (10)$$

2.9. Value of r_2 for the board at iterations $I \ge 4$
For iteration $I \ge 4$, the value of r_2 is given by adding the equations (7), (8), (9) and (10). After simplification we have the following formula

$$\sum_l \left\{ \sum_m [614 + 16(33k - 114 + 4m - 12l)] \right\} \text{ where } 1 \le m \le l, \; 1 < l \le \frac{\left(\frac{r}{2} - 7 \right)}{2} + 1 \qquad (11)$$

3. Conclusion
In this paper, a generalized formula of r_2 (number of ways of placing two non-attacking Rooks) for our Fractal Chessboard (FC) has been found by disintegrating the whole board FC into smaller boards. These smaller boards have been categorized based on their positions in FC. Since our fractal board grows consistently, the categorization of the boards was same for all the iterations and only the count of some of the categories increases for higher iterations. The paper can be further extended to three non-attacking Rooks where the formula of r_3 can be generalized for the Fractal Chessboard by enumerating the r_3 value for the categorized small boards. However, it becomes increasingly more difficult to enumerate and generalize a formula as the number of Rook increases.

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