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To cite this article: S Senthil and R Perumal 2021 *J. Phys.: Conf. Ser.* **1850** 012100

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Minimal prime ideals in seminearrings

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Abstract. This paper deals with the properties of ideals in seminearrings with special reference to right duo seminearrings. We formulate the concept of prime ideal and minimal prime ideal in a seminearring. In the case of Noetherian right duo seminearring, we derive some additional properties.

1. Introduction

In 1967, V.G. Van Hoorn and B. Van Rootselaar introduced the algebraic structure from monoids is seminearring [1]. $(S, +)$ and $(S, .)$ are semigroups with right distributive law where addition and multiplication as a binary operations is known as a right seminearring $(S, +, .)$ [2]. If for all $l \in S$, $l + 0 = 0 + l = l$ and $l.0 = 0.l = 0$ then S is an absorbing zero 0. Semigroup mapping, linear sequential machines, etc,. are the seminearring applications. $(\Gamma, +)$ is a semigroup mapping sets with absorbing zero, $\mathfrak{M}(\Gamma)$ is the form of pointwise addition and composition of mapping is the natural example of seminearring. In [3, 4] the authors studied rings(commutative) through minimal prime ideals and also in [5] vector lattices are introduced and studied over minimal prime ideal. Joseph Kist [6] studied semigroups(commutative) over minimal prime ideals. The main objective of this paper we extend the concept of minimal prime ideals to seminearrings. We furnish certain properties and investigate the results as cited [3, 4, 5] to be applied in seminearrings. In particular, we obtain the number of minimal prime ideals in Noetherian seminearrings.

This paper comprises four sections. We review some basic results and definitions about seminearrings in Section 2. In Section 3 introduce the main results of minimal prime ideals in seminearrings. We also derive the properties of minimal prime ideal in a seminearring. We concentrate Noetherian seminearrings in Section 4.

2. Preliminaries

Inside this section we compile all the terminologies that are used in our paper relating to the theory of seminearring. S has an ideal $\mathcal{I}$ is known as nilpotent if $\mathcal{I}^m = 0$ for some integer $m \geq 1$, idempotent if $\mathcal{I}^2 = \mathcal{I}$. $(S, +, .)$ is a seminearring and $\mathcal{I}$ is a non-empty subset of S is a left (respectively right) ideal of S if $(i) m + y \in \mathcal{I} \ \forall \ m, y \in \mathcal{I}$, $(ii) r.u \in \mathcal{I}(u.r \in \mathcal{I})$ $\forall \ u \in \mathcal{I} \ and \ r \in S$ conditions hold [7]. $\mathcal{I}$ is called an ideal of S if it is a both left as well as a right ideal of S. If A, B are ideals and $AB \subseteq \mathcal{P}$ gives either $A \subseteq \mathcal{P}$ or $B \subseteq \mathcal{P}$ then $\mathcal{P}$ is known as prime ideal of S. $A^2 \subseteq \mathcal{P} \implies A \subseteq \mathcal{P}$ is quasi prime ideal. A prime ideal $\mathcal{P}$ containing

an ideal A of a seminearring S is known as a minimal prime ideal belongs to A if there is no prime ideal Q such that $A \subseteq Q \subset \mathcal{P}$. If S has 0, then the minimal prime ideals belongs to 0 are called the minimal prime ideals of S. S has a subset which is non-empty K is an $M-$System if $K \cap A \neq \phi$ and $K \cap B \neq \phi \implies K \cap AB \neq \phi$ (A,B ideals of S). Noetherian seminearring is each set(non-empty) of ideals of S has a maximal element. A seminearring S is called right duo, if each right ideal of S is two sided and duo, if each one sided ideal of R is two sided. For $A \subseteq S$, we define the radical $\sqrt{A}$ of A to be $\{a \in R/a^k \in A$ for some positive integer k$\}$. Obviously $A \subseteq \sqrt{A}$.

3. Main Results

Lemma 3.1. *Let K be in M-system , not meeting in ideal A of S. Then K is contained in a M-system N , maximal as regards to the condition of not meeting A.*

Proof. Since the union of any linearly ordered set of M-system is again a M-system , we obtain the required maximal M-system N (Zorn's lemma).   □

Proposition 3.1. *Suppose S has an ideal A and K an M-system disjoint with A. Then A is included in an ideal $\mathcal{P}$, maximal as regards to the condition of not meeting K. Further $\mathcal{P}$ is prime.*

Proof. The union $\mathcal{P}$ of all ideals of S disjoint with K is clearly an ideal contains A and is maximal as regards to the condition of not meeting K.
We now cliam that $\mathcal{P}$ is prime.
For, assume $B \nsubseteq \mathcal{P}$ and $C \nsubseteq \mathcal{P}$ (B, C ideals of S).
Then $K \cap B \neq \phi$ and $K \cap C \neq \phi$ and K being an M-system, we have $K \cap BC \neq \phi$.
Suppose $BC \subseteq \mathcal{P}$. Then $K \cap BC \subseteq K \cap \mathcal{P} \neq \phi$, a contradiction.
Hence $\mathcal{P}$ is prime, proving the result.   □

Theorem 3.1. *A seminearring S has subset $\mathcal{P}$ is a minimal prime ideal including to an ideal A of S iff $\mathcal{P}'$ is a maximal $M - system$ disjoint with A.*

Proof. Let $\mathcal{P}'$ be a maximal $M - system$ in S which is disjoint with A.
Then by Proposition 3.1, $A \subseteq Q$ where Q is a prime and is maximal w.r.t the property of not meeting $\mathcal{P}'$. Since $\mathcal{P}' \cap Q = \phi$, $Q \subseteq \mathcal{P}$ and so $Q' \supseteq \mathcal{P}'$.
But $A \subseteq Q$ and so $M - system$ Q' does not meet A.
By the maximality of $\mathcal{P}'$, it follows that $Q' = \mathcal{P}'$ and so $Q = \mathcal{P}$, given $A \subseteq \mathcal{P}$.
Also there cannot be a prime ideal N such that $A \subseteq N \subset \mathcal{P}$.
For, otherwise N' is a $M - system$ not meeting A such that $N' \supset \mathcal{P}'$,
contradicting the maximality of $\mathcal{P}'$.
Hence A has a minimal prime ideal $\mathcal{P}$.
Conversely, assume A has a minimal prime ideal $\mathcal{P}$.
Then $\mathcal{P}'$ is M-system not meeting A, so by the Lemma 3.1, we can find M-system K containing $\mathcal{P}'$ and maximal as regards to the condition of not meeting A.
Which gives, K' belonging to A is a minimal prime ideal.
Since $\mathcal{P} \subseteq K$, we have $A \subseteq K' \subseteq \mathcal{P}$.
By the minimality of $\mathcal{P}$, it follows that $K' = \mathcal{P}$ and so $\mathcal{P}' = K$.
Thus $\mathcal{P}'$ is a maximal M-system disjoint with A.
This proves the theorem.   □

Corollary 3.1.1. *S has an ideal A and N is a M-system disjoint with A.*
Then A has a minimal prime ideal belonging to it.

Proof. This follows in view of Lemma 3.1 and Theorem 3.1.   □

Corollary 3.1.2. *If a prime ideal $\mathcal{P}$ contains an ideal A of a seminearring S then A has minimal prime ideal that belongs to it.*

Proof. Since $A \subseteq \mathcal{P}$, $\mathcal{P}'$ is M-system disjoint with A and so the result follows by Corollary 3.1.1   □

Corollary 3.1.3. *In a seminearring S with 0, each prime ideal contains a minimal prime ideal.*

Proof. This follows by 3.1.2.   □

Corollary 3.1.4. *In a seminearring S with 0, all minimal prime ideals intersection set is rad S.*

Proof. This follows by Corollary 3.1.3.   □

Theorem 3.2. *In a Noetherian seminearring S, for every ideal A we can find an integer $n \geq 1$ such that $[r(A)]^n \subseteq A$. Further, if S has 0, then rad S is nilpotent.*

Proof. If the result is not true,
then the set $F = [I|I$ is an ideal of S such that $[r(I)]^n \nsubseteq I$ for every integer $n \geq 1]$ is non empty.
By Noetherian condition, F contains a maximal element M.
M cannot be prime, as otherwise $[r(M)]^1 = M$, implying $M \notin F$.
Hence ideals $\mathcal{P}, Q$ exist therefore $\mathcal{P}, Q \nsubseteq M$, but $\mathcal{P}Q \subseteq M$.
Taking $C = \mathcal{P} \cup M$ and $D = Q \cup M$, we have $CD = (\mathcal{P} \cup M)(Q \cup M) \subseteq \mathcal{P}Q \cup M \subseteq M$.
Since $C, D \notin F$, find an integer K implies $[r(C)]^k \subseteq C$ and that $[r(D)]^k \subseteq D$, so that $[r(C)]^k[r(D)]^k \subseteq CD$. Since $M \subseteq C$ and D, we see that $r(M) \subseteq r(C)$ and $r(M) \subseteq r(D)$.
Hence $[r(M)]^{2k} \subseteq [r(C)]^k[r(D)]^k \subseteq CD \subseteq M$, a contradiction, as $M \in F$.
It follows that F is empty. Thus $[r(A)]^n \subseteq A$, for some integer $n \geq 1$.
If S has 0, then choosing $A = 0$ we conclude that $rad\ S = r(0)$ is nilpotent.   □

Corollary 3.2.1. *If S is Noetherian then S has ideals A and B, $A \subseteq r(B)$ iff $A^n \subseteq B$.*

Proof. Let $A \subseteq r(B)$. By Theorem 3.2, $[r(B)]^n \subseteq B$ for some integers $n \geq 1$ and so $A^n \subseteq B$. Conversely, if for some integer $n \geq 1$ we have $A^n \subseteq B$, then $A^n \subseteq r(B)$ and since $r(B)$ is quasi prime, we get $A \subseteq r(B)$.
This proves result.   □

Proposition 3.2. *Let S be a Noetherian right duo seminearring with 0. Then rad S is the unique maximal nilpotent ideal of S.*

Proof. In view of the Theorem 3.2, it is enough to prove that every nilpotent ideal of S is contained in $rad\ S$. Let A be a nilpotent ideal of S .Then $A^n = 0$ for some integers $n \geq 1$.
For $x \in A$, we have
$x^n \in A^n = 0 \subseteq r(0)$
$\implies (x)^n \subseteq (0)$ (S being right duo)
$\implies (x) \subseteq r(0)$ ($r(0)$ being quasi prime)
$\implies (x) \in r(0)$, so that $A \subseteq rad\ S$.
This proves the result   □

Theorem 3.3. *In a Noetherian seminearring S, every ideal A has almost a finite number of minimal prime ideals belonging to it.*

Proof. Suppose the result does not hold.

Then by Noetherian condition, we can find an ideal M, maximal among those for which the result does not hold. Clearly M cannot be prime and so ideals B, C exist in S hence $B \nsubseteq M$, $C \nsubseteq M$, but $BC \subseteq M$.

Taking $D = B \cup M$ and $E = C \cup M$, we see that $DE = (B \cup M)(C \cup M) \subseteq BC \cup M \subseteq M$.

Since D, E contains M properly, by maximality of M, we see that each of D and E has a finite number of minimal prime ideals that belongs to it.

Assume M contains a minimal prime ideal $\mathcal{P}$. Then $DE \subseteq M \subseteq \mathcal{P}$, so that $D \subseteq \mathcal{P}$ or $E \subseteq \mathcal{P}$. Thus $M \subset D \subseteq \mathcal{P}$ or $M \subset E \subseteq \mathcal{P}$ and so $\mathcal{P}$ is a minimal prime ideal belongs to either D or E. Hence M has finite number of minimal prime ideals - a contradiction.

This proves the theorem. $\square$

Corollary 3.3.1. *A Noetherian seminearring S with 0 has minimal prime ideals which is finite.*

4. Conclusion

This makes the interest to study the minimal prime ideals and its important properties of seminearrings. We observe some fruitful results of minimal prime ideals for the case of Noetherian right duo seminearring.

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