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MEAN LABELING FOR STAR AND PATH GRAPH

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ABSTRACT. In this paper, we proved that for $\alpha \leq \beta$, the graph $K_{1,\alpha} \wedge P_\beta$ is a mean graph if $\alpha \leq 6$; $\beta > 1$ and not a mean graph if $\alpha \geq 9$; $\beta > 1$.

1. INTRODUCTION

Somasundaram and Ponraj [8] have introduced the notion of mean labelings of graphs. A graph G with p nodes and q links is called a mean graph if there is an one to one function φ from the nodes of G to $\{0, 1, 2, \cdots q\}$ such that when each link $\lambda\mu$ is labeled with $\frac{(\varphi(\lambda)+\varphi(\mu))}{2}$ if $\varphi(\lambda) + \varphi(\mu)$ is even, and $\frac{(\varphi(\lambda)+\varphi(\mu)+1)}{2}$ if $\varphi(\lambda) + \varphi(\mu)$ is odd, then the resulting link labels $\{1, 2, \cdots q\}$ are distinct. In [1], [2], [8], [9], [10], [11], [4], and [5] they proved the following graphs are mean graphs: P_β; C_β; $K_{2,\alpha}$; $K_2 + mK_1$; $\overline{K_n} + 2K_2$; $C_\alpha \cup P_\beta$; $P_\alpha \times P_\beta$; $P_\alpha \times C_\beta$; $C_\alpha \odot K_1$; $P_\beta \odot K_1$, triangular snakes, quadrilateral snakes, K_α if and only if $\alpha < 3$; $K_{1,\alpha}$ if and only if $\alpha < 3$, bistars $B_{\alpha,\beta}$ $(\alpha > \beta)$ if and only if $\alpha < \beta + 2$, the subdivision graph of the star $K_{1,\alpha}$ if and only if $\alpha < 4$. In [3], they proved that two star $K_{1,\alpha} \wedge K_{1,\beta}$ is a mean graph if and only if $|\alpha - \beta| \leq 4$. In [6], they proved that if $\alpha \leq \beta < \gamma$, then the three star $K_{1,\alpha} \wedge K_{1,\beta} \wedge K_{1,\gamma}$ is a mean graph if and only if $\alpha + \beta - 8 \leq \gamma \leq \alpha + \beta + 4$ when $\alpha > 9$ and $\beta + 1 \leq \gamma \leq \alpha + \beta + 4$ when $1 \leq \alpha \leq 9$. In [7], they have proved that the four star $K_{1,g_1} \wedge K_{1,g_2} \wedge K_{1,g_3} \wedge K_{1,g_4}$, $g_1 \leq g_2 \leq g_3 \leq g_4$ is a mean graph if and only

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if
1. $g_4 \in [g_3, g_1 + g_2 + g_3 + 5]$ if $g_1 \leq 9$ and
2. $g_4 \in [g_3, g_3 + 7] \cup [g_1 + g_2 + g_3 - 11, g_1 + g_2 + g_3 + 5]$ if $g_1 > 9$.

By refering these results we got motivated and worked on Mean labeling for star and path graphs.

Definition 1.1. *A wedge is a link which is used for connecting two components of a graph. It is denoted as $\wedge$, $\lambda(G\wedge) < \lambda(G)$. where λ denotes the number of components of graph.*

Definition 1.2. *A graph G with p nodes and q links is called a mean graph if there is an one to one function φ from the nodes of G to $\{0, 1, 2, \cdots q\}$ such that when each link $\lambda\mu$ is labeled with $\frac{(\varphi(\lambda)+\varphi(\mu))}{2}$ if $\varphi(\lambda) + \varphi(\mu)$ is even, and $\frac{(\varphi(\lambda)+\varphi(\mu)+1)}{2}$ if $\varphi(\lambda) + \varphi(\mu)$ is odd, then the resulting link labels $\{1, 2, \cdots q\}$ are distinct.*

2. Main Results

Theorem 2.1. *For $\alpha \leq \beta$, the graph $K_{1,\alpha} \wedge P_\beta$ is mean graph if $\alpha \leq 6$ and $\beta > 1$.*

Proof. Let the graph $G = K_{1,\alpha} \wedge P_\beta$. Let $\alpha \leq \beta$. Let $\{\rho\} \cup \{\rho_i : 1 \leq i \leq \alpha\}$; $\{\kappa_j : 1 \leq j \leq \beta\}$ be the nodes of G. Then We have,

$$V(G) = \{\rho\} \cup \{\rho_i : 1 \leq i \leq \alpha\} \cup \{\kappa_j : 1 \leq j \leq \beta\}$$

and

$$E(G) = \{\rho\rho_i : 1 \leq i \leq \alpha\} \cup \{\kappa_j\kappa_{j+1} : 1 \leq j \leq \beta - 1\} \cup \{\rho_i\kappa_j : \text{ for some } i \text{ and } j\}.$$

Then G has $\alpha + \beta + 1$ nodes and $\alpha + \beta$ links. Node and link labeling of G is given by the function φ anf φ^* respectively, where, $\varphi : V(G) \to \{0, 1, 2, \cdots q = \alpha + \beta\}$ and $\varphi^* : E(G) \to \{1, 2, \cdots q = \alpha + \beta\}$. Now we have to prove that G is mean graph if $\alpha \leq 6$ and $\beta > 1$.

Case 1 When $\alpha = 1, 2, 3$ and $\beta > 1$. The node labeling of G:

$$\varphi(\rho) = q - 1;\ \varphi(\rho_1) = q;$$

$$\varphi(\rho_2) = q - 2;\ \varphi(\rho_3) = \varphi(\rho_2) - 1$$

and

$$\varphi(\kappa_j) = j - 1 \text{ for } 1 \leq j \leq \beta.$$

The link labeling of G: $\varphi^*(\rho\rho_1) = q$; $\varphi^*(\rho\rho_2) = q - 1$; $\varphi^*(\rho\rho_3) = q - 2$ and $\varphi^*(\kappa_j\kappa_{j+1}) = j$ for $1 \leq j \leq \beta - 1$.

The wedge $\rho_\alpha\kappa_\beta$ is $\varphi^*(\kappa_{\beta-1}\kappa_\beta)+1$. Hence all the node and link labels are distinct. Therefore G is mean graph when $\alpha=1,2,3$ and $\beta>1$.

Case 2 When $\alpha=4$ and $\beta>1$. The node labeling of G:

$$\varphi(\rho)=q-1; \varphi(\rho_1)=q;\ \varphi(\rho_2)=q-2;$$

$$\varphi(\rho_3)=\varphi(\rho_2)-1;\ \varphi(\rho_4)=\varphi(\rho_3)-2;$$

$$\varphi(\kappa_j)=j-1 \text{ for } 1\leq j\leq \beta-1 \text{ and } \varphi(\kappa_\beta)=\varphi(\kappa_{\beta-1})+2.$$

The link labeling of G: $\varphi^*(\rho\rho_1)=q$; $\varphi^*(\rho\rho_i)=\varphi^*(\rho\rho_{i-1})-1$ for $i=2,3,4$ and $\varphi^*(\kappa_j\kappa_{j+1})=j$ for $1\leq j\leq\beta-1$. The wedge $\rho_4\kappa_\beta$ is $\varphi^*(\kappa_{\beta-1}\kappa_\beta)+1$.

Hence all the node and link labels are distinct. Therefore G is mean graph when $\alpha=4$ and $\beta>1$.

Case 3 When $\alpha=5$ and $\beta>1$. The node labeling of G:

$$\varphi(\rho)=q-1; \varphi(\rho_1)=q;\ \varphi(\rho_2)=q-2; \varphi(\rho_3)=\varphi(\rho_2)-1;$$

$$\varphi(\rho_i)=\varphi(\rho_{i-1})-2 \text{ for } i=4,5;\ \varphi(\kappa_j)=j-1 \text{ for } 1\leq j\leq\beta-2$$

and

$$\varphi(\kappa_j)=\varphi(\kappa_{j-1})+2 \text{ for } j=\beta-1,\beta.$$

The link labeling of G:

$$\varphi^*(\rho\rho_1)=q;\ \varphi^*(\rho\rho_i)=\varphi^*(\rho\rho_{i-1})-1 \text{ for } i=2,3,4,5;$$

$$\varphi^*(\kappa_j\kappa_{j+1})=j \text{ for } 1\leq j\leq\beta-2; \text{ and } \varphi^*(\kappa_{\beta-1}\kappa_\beta)=\varphi^*(\kappa_{\beta-2}\kappa_{\beta-1})+2.$$

The wedge $\rho_5\kappa_{\beta-1}$ is $\varphi^*(\kappa_{\beta-2}\kappa_{\beta-1})+1$.

Hence all the node and link labels are distinct. Therefore G is mean graph when $\alpha=5$ and $\beta>1$.

Case 4 When $\alpha=6$ and $\beta>1$. The node labeling of G:

$$\varphi(\rho)=q-1; \varphi(\rho_1)=q;\ \varphi(\rho_2)=q-2;\ \varphi(\rho_3)=\varphi(\rho_2)-1;$$

$$\varphi(\rho_i)=\varphi(\rho_{i-1})-2 \text{ for } i=4,5;\ \varphi(\rho_6)=\varphi(\rho_5)-4;$$

$$\varphi(\kappa_j)=j-1 \text{ for } 1\leq j\leq\beta-5; \varphi(\kappa_{\beta-4})=\varphi(\kappa_{\beta-5})+2;$$

$$\varphi(\kappa_j)=j \text{ for } j=\beta-3,\beta-2$$

and

$$\varphi(\kappa_j)=\varphi(\kappa_{j-1})+2 \text{ for } j=\beta-1,\beta.$$

The link labeling of G:

$$\varphi^*(\rho\rho_1) = q;\ \varphi^*(\rho\rho_i) = \varphi^*(\rho\rho_{i-1}) - 1 \text{ for } i = 2,3,4,5;$$

$$\varphi^*(\rho\rho_6) = \varphi^*(\rho\rho_5) - 2;\ \varphi^*(\kappa_j\kappa_{j+1}) = j \text{ for } 1 \le j \le \beta - 6;$$

$$\varphi^*(\kappa_{\beta-5}\kappa_{\beta-4}) = \varphi^*(\kappa_{\beta-6}\kappa_{\beta-5}) + 1;\ \varphi^*(\kappa_{\beta-4}\kappa_{\beta-3}) = \varphi^*(\kappa_{\beta-5}\kappa_{\beta-4}) + 2;$$

$$\varphi^*(\kappa_j\kappa_{j+1}) = \varphi^*(\kappa_{j-1}\kappa_j) + 1 \text{ for } j = \beta-3, \beta-2$$

and $\varphi^*(\kappa_{\beta-1}\kappa_\beta) = \varphi^*(\kappa_{\beta-2}\kappa_{\beta-1}) + 2$. The wedge $\rho_6\kappa_{\beta-4}$ is $\varphi^*(\kappa_{\beta-5}\kappa_{\beta-4}) + 1$.

Hence all the node and link labels are distinct. Therefore G is mean graph when $\alpha = 6$ and $\beta > 1$. Hence, For $\alpha \le \beta$, the graph $K_{1,\alpha} \wedge P_\beta$ is mean graph if $\alpha \le 6$. □

Remark 2.1. *For the graphs $K_{1,7} \wedge P_\beta$ and $K_{1,8} \wedge P_\beta$, $\beta > 1$ the labeling exists if $\alpha \le \beta$ but we are not able to generalise it.*

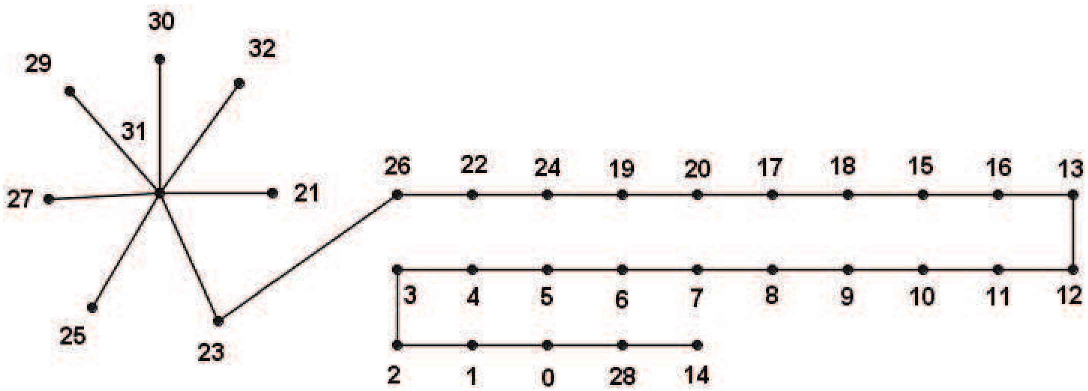

FIGURE 1. $K_{1,7} \wedge P_{25}$

Theorem 2.2. *For $\alpha \le \beta$, the graph $K_{1,\alpha} \wedge P_\beta$ is not a mean graph if $\alpha \ge 9$ and $\beta > 1$.*

Proof. Let $G = K_{1,\alpha} \wedge P_\beta$. Let $\alpha \le \beta$. Now we have to prove that G is not a mean graph if $\alpha \ge 9$ and $\beta > 1$. Suppose G is a mean graph for $\alpha \ge 9$ and $\beta > 1$. Let

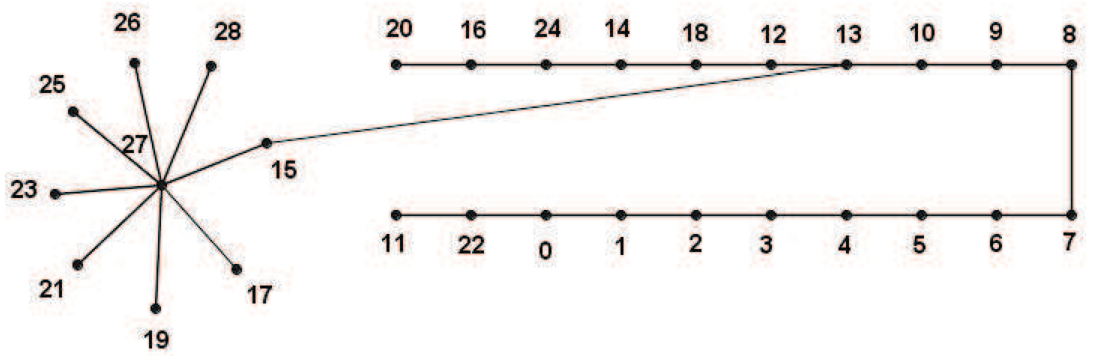

FIGURE 2. $K_{1,8} \wedge P_{20}$

us now consider the case when $\alpha = 9$ and $\beta = 9$ then G has 19 nodes and 18 links. we define,

$$V(G) = \{\rho\} \cup \{\rho_i : 1 \leq i \leq \alpha = 9\} \cup \{\kappa_j : 1 \leq j \leq \beta = 9\}$$

and

$$E(G) = \{\rho\rho_i : 1 \leq i \leq \alpha = 9\} \cup \{\kappa_j\kappa_{j+1} : 1 \leq j \leq \beta - 1 = 8\}.$$

Suppose G is a mean graph and let $p = 19$ and $q = 18$. Then there is an one to one function φ from the nodes of G to $\{0, 1, 2, \cdots q\}$ such that when each link $\lambda\mu$ is labeled with $\frac{(\varphi(\lambda)+\varphi(\mu))}{2}$ if $\varphi(\lambda) + \varphi(\mu)$ is even, and $\frac{(\varphi(\lambda)+\varphi(\mu)+1)}{2}$ if $\varphi(\lambda) + \varphi(\mu)$ is odd, then the resulting link labels $\{1, 2, \cdots q\}$ are distinct. Let us first label nodes of $K_{1,\alpha}$ such that its links are labeled as $\{10, 11, 12, \cdots 18\}$. The possibility of the link label 18 are 17 and 18. Therefore 18 should be the label of the graph G. Let it be $\varphi(\rho) = 18$. The possibilities for $\varphi(\rho_i)$, for $1 \leq i \leq 9$ for obtaining the link labels $\{10, 11, 12, \cdots 18\}$ are 17, (16 or 15), (14 or 13), (12 or 11), (10 or 9), (8 or 7), (6 or 5), (4 or 3), (2 or 1) which implies $\varphi(\rho_1) = 17$. Now we label the nodes of P_9 such that its links are among $\{1, 3, 4, 5, 6, 7, 8, 9\}$ and 2 be the wedge label.

Case 1: The possibilities to get link label 1 are (0 and 1) or (0 and 2). Let $\varphi(\kappa_2) = 0$ then, $\varphi(\rho_9)$ should be either 1 or 2. Suppose $\varphi(\rho_9) = 1$ then 2 should be the label of any one of the nodes of P_9. Let $\varphi(\kappa_3) = 2$ then $\varphi^*(\kappa_2\kappa_3) = 1$.

Case 2: consider $\varphi(\rho_8)$ should be either 3 or 4. If $\varphi(\rho_8) = 3$ then 4 will be the label of any one of the nodes of P_9. If we assume $\varphi(\kappa_4) = 4$ then $\varphi^*(\kappa_3\kappa_4) = 3$.

Case 3: Let us fix $\varphi(\rho_7)$ should be either 5 or 6. In case $\varphi(\rho_7) = 5$ then 6 should be the label of any one of the nodes of P_9. Let $\varphi(\kappa_5) = 6$ then $\varphi^*(\kappa_4\kappa_5) = 5$.

Case 4: Let us set $\varphi(\rho_6)$ should be either 7 or 8. Suppose $\varphi(\rho_6) = 7$ then 8 must be the label of any of the nodes of P_9. Let $\varphi(\kappa_6) = 8$ then $\varphi^*(\kappa_5\kappa_6) = 7$.

Case 5: consider $\varphi(\rho_5)$ should be either 9 or 10. If $\varphi(\rho_5) = 9$ then 10 should be the label of any of the nodes of P_9. Assume $\varphi(\kappa_7) = 10$ then $\varphi^*(\kappa_6\kappa_7) = 9$.

Case 6: Let us set $\varphi(\rho_4)$ should be either 11 or 12. In case $\varphi(\rho_4) = 11$ then 12 must be the label of any of the nodes of P_9. Now Consider $\varphi(\kappa_8) = 12$ then $\varphi^*(\kappa_7\kappa_8) = 11$ (which is not possible). Therefore, $\varphi(\kappa_8) \neq 12$. Let $\varphi(\kappa_1) = 12$ then $\varphi^*(\kappa_1\kappa_2) = 6$.

Case 7: Let us fix $\varphi(\rho_3)$ should be either 13 or 14. Suppose $\varphi(\rho_3) = 13$ then 14 should be the label of any of the nodes of P_9. Let us assume that $\varphi(\kappa_8) = 14$ then $\varphi^*(\kappa_7\kappa_8) = 12$ (which is not possible). Therefore, $\varphi(\kappa_8) \neq 14$. suppose $\varphi(\kappa_8) = 13$ then $\varphi^*(\kappa_7\kappa_8) = 12$ (which is not possible). Therefore, $\varphi(\kappa_8) \neq 13$. since $\varphi(\rho_2)$ should be either 14 or 15. If we assume $\varphi(\kappa_8) = 14$ then $\varphi^*(\kappa_7\kappa_8) = 13$ (which is not possible). Therefore, $\varphi(\kappa_8) \neq 14$. suppose $\varphi(\kappa_8) = 15$ then $\varphi^*(\kappa_7\kappa_8) = 13$ (which is not possible). Therefore, $\varphi(\kappa_8) \neq 15$. The remaining possible link labels are $\{4, 8\}$ which is not possible when $\varphi(\kappa_j) =$ 13 or 14 or 15 or 16. Therefore, G is not a mean graph when $\alpha = \beta = 9$. Similarly we can prove for all other cases. Therefore, G is not a mean graph when $\alpha \geq 9$ and $\beta > 1$ for $\alpha \leq \beta$. Therefore, $K_{1,\alpha} \wedge P_\beta$ is not a mean graph if $\alpha \geq 9$ and $\beta > 1$ for $\alpha \leq \beta$. □

3. CONCLUSION

In this paper we proved that for $\alpha \leq \beta$, the graph $K_{1,\alpha} \wedge P_\beta$ is a mean graph if $\alpha \leq 6$; $\beta > 1$ and not a mean graph if $\alpha \geq 9$; $\beta > 1$. In future, we planned to work on Mean labeling for two star and a path graph.

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