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FIXED POINTS-JULIA SETS

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ABSTRACT. In this paper, the fixed points of a polynomial are discussed. The fixed points form the fractals. The Julia sets formed due to the iteration of the fixed points which are fractal sets.

1. INTRODUCTION

The main content of this paper is the description and application of fixed point theory in a metric space. The term fractal [1] was first used by mathematician Benoit Mandelbrot in 1975. Mandelbrot based it on the Latin fractus meaning broken or fractured, and it extends to the concept of theoretical fractional dimensions to geometric patterns in nature.

Fractal is the mathematical set that exhibits a repeating pattern displayed at every scale [4]. It is also known as expanding symmetry or evolving symmetry. If the replication is exactly the same at every scale, then it is called a self-similar pattern. Examples are Cantor sets, Sierpinski triangle and Menger Sponge etc. The study of complex dynamical systems deals with the local behaviour near fixed points.

Banach contraction principle [5] is of more importance in fixed point theory with a wide range of applications, which includes iterative methods for solving linear, non-linear, differential, integral, and difference equations.

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2. PRELIMINARIES

2.1. **Fixed Point.** A fixed point is a point x such that $f(x) = x$. The points in the graph of f, $y = f(x)$, crosses the diagonal, whose equation is $y = x$.

2.2. **METRIC SPACE.** Let X be a non-empty set. Then the ordered pair (X, d) is called a metric space if d is a metric for X and d(x, y) is called the distance between x and y for x, y $\in$ X.

A mapping $d : X \times X \to R$ is said to be a metric if and only if the following conditions hold:

- d(x,y) $\geq$ 0 for all x, y $\in$ X;
- d(x,y) = 0 if and only if x = y;
- d(x,y) = d(y, x) for all x, y $\in$ X;
- d(x,y) $\leq$ d(x, z) + d(z,y) for all x, y, z $\in$ X.

Let (X, d) be a metric space and let A be non-empty subset of X. Then the diameter of A, denoted by δ(A), is defined by δ(A) = sup {d(x,y): x, y $\in$ A} that is, the diameter of A is the supremum of the set of all distance between points of A.

A sequence of elements $x_1, x_2, \ldots, x_n, \ldots$ in a metric space X is said to converge to an element x $\in$ X if the sequence of real numbers $d(x_n, x)$ converges to zero as $n \to \infty$.

2.3. **CAUCHY SEQUENCE.** A sequence of points p_n in X is said to be a Cauchy sequence in X if and only if for every $\varepsilon > 0$ there exists a positive integer $n(\varepsilon)$, such that for m, n $\geq$ n (ε) $\Rightarrow$ $d(p_m, p_n) < \varepsilon$ [2].

2.4. **COMPLETE SEQUENCE.** Every convergent sequence in a metric space is a Cauchy sequence. A metric space (X, d) is said to be complete if and only if every Cauchy sequence in X converges.

2.5. **CONTRACTION MAPPING.** Given a metric space X with metric d, a mapping $S : X \to X$ is a contraction if $d(s(x), s(y)) \leq cd(x, y)$ for some c < 1 and for all x, y $\in$ X. If equality holds everywhere, then we call S a contracting similarity [3].

2.6. **BANACH SPACE.** A Banach space is a complete normed vector space with a metric that allows the computation of vector length and distance between vectors is complete in the sense that a cauchy sequence of vector always converges to a well defined limit that is within the space.

2.7. **LIPSCHITZ MAPPING.** A self mapping f of a metric (X, d) is said to be Lipschitz mapping if for all x, y ∈ X and some $\alpha \geq 0$, it holds that:

$$d(f(x), f(y)) \leq \alpha d(x, y).$$

The mapping f is said to be contraction if the above equation holds for some $\alpha \in [0, 1)$, and non-expansive if $\alpha = 1$. A contraction mapping is always continuous.A new contraction called F-contraction is introduced and generalization of the Banach contraction principle results in a fixed point. In a complete metric space (X, d), every F - contractive self-map has a unique fixed point in X and for every x_0 in X a sequence of iterates $x_0, fx_0, f^2x_0, \ldots$ converges to the fixed point of f.

2.8. **JULIA SET.** The Julia Set is dependent upon the complex numbers which have both a real and imaginary component ι. Julia set is a set of complex numbers which do not converge to any limit when a given mapping is repeatedly applied to it.

Theorem 2.1. *Let f be a contraction on a complete metric space X. Then f has a unique fixed point $x \in X$.*

Proof. If $x_1, x_2 \in X$ are fixed points of f, then

$$\mathrm{d}\left(\mathrm{x}_1, \mathrm{x}_2\right) = \mathrm{d}\left(f\left(\mathrm{x}_1\right), \mathrm{d}\left(f\left(\mathrm{x}_2\right),\right) \leq \lambda \mathrm{d}\left(\mathrm{x}_1, \mathrm{x}_2\right),$$

which implies $x_1 = x_2$.

For any $x_0 \in X$, define the iterate sequence $\mathrm{x}_{\mathrm{n}+1} = f\left(\mathrm{x}_{\mathrm{n}}\right)$. By the induction on n,

$$\begin{aligned} \mathrm{d}\left(\mathrm{x}_{\mathrm{n}+1,}\mathrm{x}_{\mathrm{n}}\right) &\leq \lambda^{\mathrm{n}}\mathrm{d}\left(f\left(\mathrm{x}_{0}\right),\mathrm{x}_{0}\right) \\ \mathrm{d}\left(\mathrm{x}_{\mathrm{n}},\mathrm{x}_{\mathrm{m}}\right) &\leq \sum_{\mathrm{i}=\mathrm{n}+1}^{\mathrm{m}} d\left(x_{i},x_{i}-1\right) \\ &\leq \sum_{\mathrm{i}=\mathrm{n}+1}^{\mathrm{m}} c^{i-1}d\left(x_{1},x_{0}\right) \\ &\leq \frac{c^{n}}{1-c}d\left(x_{1},x_{0}\right) \end{aligned}$$

Since $c < 1$, $d(x_n, x_m)$ approaches 0 for large n, and so $\{x_n\}$ is a cauchy sequence, since X is complete, $\{x_n\}$ converges to some $\mathrm{x} = \lim_{n\to\infty} \mathrm{x_n}$ in X. S is continuous since it is contraction. Therefore,

$$\begin{aligned} S(x) &= S\left(\lim_{n\to\infty} x_n\right) \\ &= \lim_{n\to\infty} S\left(x_n\right) \\ &= \lim_{n\to\infty} x_{n+1} \\ &= X \end{aligned}$$

Uniqueness of x: Suppose there exists a second fixed point y. Then, from

$$\mathrm{d}(\mathrm{s}(\mathrm{x}),\mathrm{s}(\mathrm{y})) \leq \mathrm{cd}(\mathrm{x},\mathrm{y})$$

since $\mathrm{s}(\mathrm{x}) = \mathrm{x}$ and $\mathrm{s}(\mathrm{y}) = \mathrm{y}$, we have

$$\mathrm{d}(\mathrm{x},\mathrm{y}) \leq \mathrm{cd}(\mathrm{x},\mathrm{y}),$$

i.e., $\mathrm{d}(\mathrm{x},\mathrm{y}) = 0$ or $\mathrm{x} = \mathrm{y}$

For example, The fixed points of the function $f(x) = x^3$,

$$x^3 = x \quad \Rightarrow \quad x^3 - x = 0 \quad \Rightarrow \quad x\left(x^2 - 1\right) = 0.$$

So the fixed points are $x = 0$ and $x = -1$, $x = +1$.

□

The fixed points create Julia sets which are fractal sets. Hence proved.

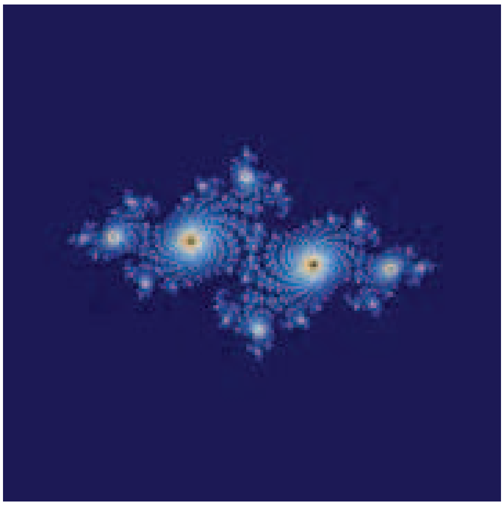

FIGURE 1. **Julia set for** $f_c, c = -0.7269 + 0.1889i$

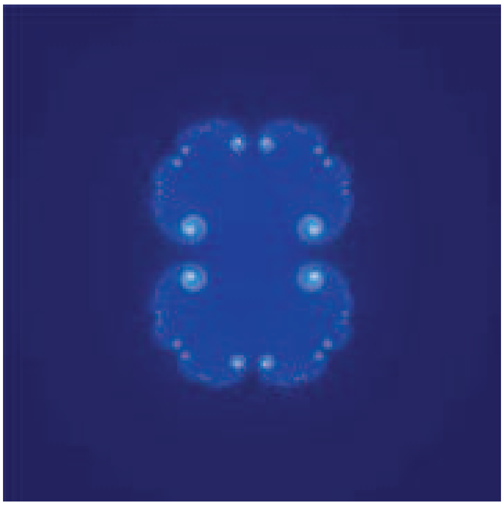

FIGURE 2. **Julia set for** $f_c, c = 0.285 + 0i$

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