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CORONA PRODUCT OF GRACEFUL TREES

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ABSTRACT. A function f is called graceful labeling of a graph G with m edges, if f is an injective function from $V(G)$ to $\{0, 1, 2, \dots, m\}$ such that, when every edge uv is assigned the edge label $|f(u) - f(v)|$, then the resulting edge labels are distinct. A graph which admits graceful labeling is called a graceful graph. The fifty-year old Graceful Tree Conjecture, due to Rosa, Ringel and Kotzig states that every tree is graceful. Let G and H be two graphs and let n be the order of G. The corona product, or simply the corona, of graphs G and H is the graph $G \odot H$ obtained by taking one copy of G and n copies of H and then joining by an edge the ith vertex of G to every vertex in the ith copy of H. Note that when G is a tree and $H \cong K_1$, the corona $G \odot K_1$ is also a tree. In this note, we prove that if T is a graceful tree, then the corona $T \odot K_1$ is also graceful.

1. INTRODUCTION

All the graphs considered in this paper are finite simple graphs. Terms that are not defined here can be referred from the book [12]. A function f is called graceful labeling of a graph G with m edges, if f is an injective function from $V(G)$ to $\{0, 1, 2, \cdots, m\}$ such that, when every edge uv is assigned the edge label $|f(u) - f(v)|$, then the resulting edge labels are distinct. A graph which admits graceful labeling is called a graceful graph. The long-standing Ringel-Kotzig-Rosa Conjecture [5, 9, 10] popularly called Graceful Tree Conjecture, which states that "All trees are graceful". The graceful tree conjecture has been the

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focus of many papers for over four decades. In the absence of a generic proof, one approach used in investigating the Graceful Tree Conjecture is proving the gracefulness of specialized classes of trees. In this direction, Mavronicolas and Michael [8] proved a substitution theorem for graceful trees, which enables the construction of a larger graceful tree through combining smaller and not necessarily identical graceful trees. Koh et al. [6, 7] proved that rooted product of graceful trees are graceful. Si-Zhang Liu et al. [11] proved that the radical product of graceful trees are graceful. Burzio and Ferrarese [1] proved the subdivision graph of a graceful tree is a graceful tree. Hrnčiar et al. [4] proved that all trees of diameter five are graceful. For an exhaustive survey on Graceful Tree Conjecture, refer the excellent survey by Gallian [3]. In this note, we prove that if T is a graceful tree, then the corona $T \odot K_1$ is also graceful.

2. Corona Product of Graphs

Let G and H be two graphs and let n be the order of G. The corona product, or simply the corona, of graphs G and H is the graph $G \odot H$ obtained by taking one copy of G and n copies of H and then joining by an edge the ith vertex of G to every vertex in the ith copy of H. Given a vertex $g \in G$, the copy of H connected to g is denoted by H_g [2]. Complete graphs, stars and wheels are basic examples of corona product families. Next we list some basic properties of corona products of graphs G and H, whose proofs are direct consequences of the definition:

- $|V(G \odot H)| = |V(G)|(|V(H) + 1)$.
- The graph $G \odot H$ is connected if and only if G is connected.
- The graph $G \odot H$ is complete if and only if $G \cong K_1$ and H is complete.
- The corona product is neither associative nor commutative.
- If G is connected, then diam$(G \odot H)$= diam(G)+2.

Observation 1: When G is a tree T with m edges and $H \cong K_1$, the corona $T \odot K_1$ is also a tree with $2m + 1$ edges. Thus, the number of newly added vertices in the corona product of T and K_1 will be $m + 1$ and all of those vertices are of degree 1. An example of a tree T with 16 edges is shown in Figure 1 and its corona $T \odot K_1$ is shown in Figure 2.

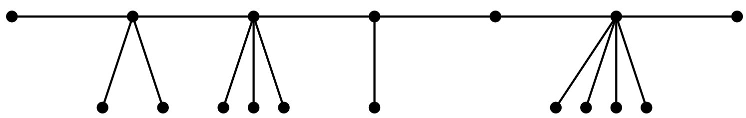

FIGURE 1. Tree T with 16 edges

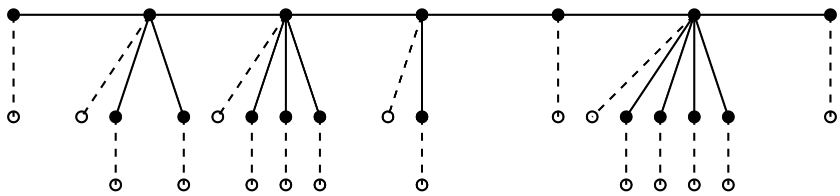

FIGURE 2. Corona Tree $T \odot K_1$ with 33 edges

Notation: For the sake of convenience, let $V(K_1) = \{w\}$ and if u be any vertex in the tree T, then the corresponding vertex added in the corona tree $T \odot K_1$ will be denoted as w_u.

3. LABELING FUNCTION

Let f be a graceful labeling of the tree T. Let $V(T) = \{u_0, u_1, \cdots, u_m\}$ and $V(K_1) = \{w\}$. Therefore, $V(T \odot K_1) = V_1 \cup V_2$, where $V_1 = \{u_0, u_1, \cdots, u_m\}$ and $V_2 = \{w_{u_0}, w_{u_1}, \cdots, w_{u_m}\}$. We will define the labeling function $\varphi : V(T \odot K_1) \to \{0, 1, 2, \cdots, 2m+1\}$ for the corona $T \odot K_1$.

First let us label the vertices in V_2 of $T \odot K_1$ as follows:

$$\varphi(w_{u_i}) = 2f(u_i), \text{ for } 0 \leq i \leq m.$$

Now, let us label the vertices in V_1 of $T \odot K_1$ as follows:

$$\varphi(u_i) = (2m+1) - 2f(u_i) = (2m+1) - \varphi(w_{u_i}), \text{ for } 0 \leq i \leq m.$$

Theorem 3.1. *The vertex labels of the corona $T \odot K_1$ are distinct.*

Proof. By the definition of labeling function φ, it is clear that the labels of vertices in V_1 are odd and the labels of vertices in V_2 are even. Since f is a graceful labeling, the vertex labels of the corona $T \odot K_1$ are distinct. $\square$

Theorem 3.2. *The edge labels of the corona $T \odot K_1$ are distinct.*

Proof. By the definition of corona product of tree T and K_1, the edges of corona tree $E(T \odot K_1) = E_1 \cup E_2$ where E_1 is the set of edges whose both the incident vertices are in V_1 and E_2 is the set of edges whose one end vertex in V_1 and the other end vertex in V_2. Further observe that by the definition of corona product, no edges in $T \odot K_1$ whose both the incident vertices are in V_2.

Since the edges in E_1 that have both the incident vertex are in V_1 and the labels of vertices in V_1 are odd, the labels of edges in E_1 are even. Similarly, the edges in E_2 that have one incident vertex in V_1 and the other incident vertex in V_2, the labels of edges in E_2 are odd.

Claim 1: The labels of edges in E_1 are distinct.

To prove Claim 1, we assume the contrary that there exists two edges $e_1 = u_1u_2$ and $e_2 = u_3u_4$ in E_1 whose edge labels are equal and whose incident vertices are different. Then, we have $|\varphi(u_2) - \varphi(u_1)| = |\varphi(u_4) - \varphi(u_3)|$. Using (2) we have, $|(2m+1) - 2f(u_2) - [(2m+1) - 2f(u_1)]| = |(2m+1) - 2f(u_4) - [(2m+1) - 2f(u_3)]|$. On simplifying, we have $|f(u_2) - f(u_1)| = |f(u_4) - f(u_2)|$, a contradiction to the fact that f is a graceful labeling. Thus, the labels of edges in E_1 are distinct and are defined from the set $\{2, 4, 6, \cdots, 2m\}$.

Claim 2: The labels of edges in E_2 are distinct.

To prove Claim 2, we assume the contrary that there exists two edges $e_1 = u_1w_{u_1}$ and $e_2 = u_2w_{u_2}$ in E_2 whose edge labels are equal and whose incident vertices are different. Then, we have $|\varphi(u_1) - \varphi(w_{u_1})| = |\varphi(u_2) - \varphi(w_{u_2})|$. Using (1) and (2) we have, $|(2m+1) - f(u_1) - 2f(u_1)]| = |(2m+1) - f(u_2) - 2f(u_2)|$. On simplifying, we have $f(u_1) = f(u_2)$, which is possible only if u_1 and u_2 are one and the same vertices, a contradiction to the fact that the incident vertices are different. Thus, the labels of edges in E_2 are distinct and are defined from the set $\{1, 3, 5, \cdots, 2m+1\}$.

From claims 1 and 2, it is clear that the edge labels of the corona $T \odot K_1$ are distinct □

Theorem 3.3. *The corona product $T \odot K_1$ of graceful tree T with m edges is graceful.*

Proof. It follows from Theorems 3.1 and 3.2. □

4. EXAMPLE

In this section, we give example of a graceful tree T in Figure 3 and the gracefulness of corona $T \odot K_1$ in Figure 5.

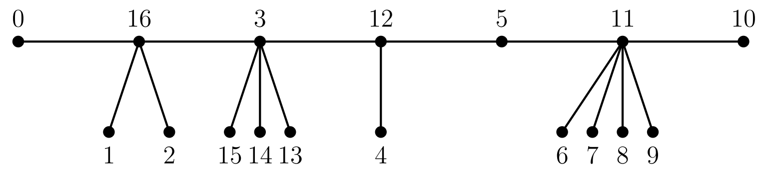

FIGURE 3. Graceful tree T with 16 edges

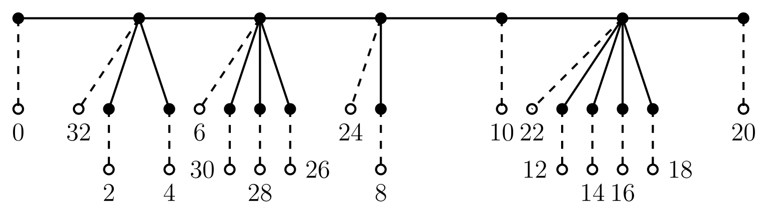

FIGURE 4. Corona Tree $T \odot K_1$ with all even labeled vertices as defined in Equation 1

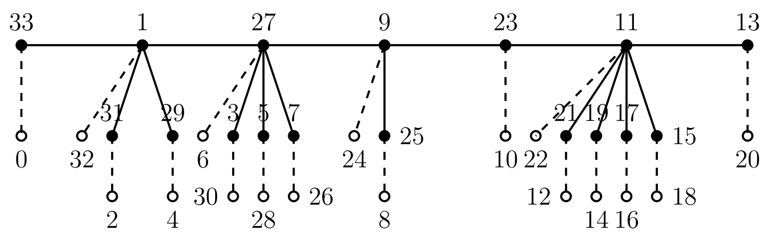

FIGURE 5. Gracefully labeled corona tree $T \odot K_1$ with 33 edges

5. CONCLUSION

We proved that if T is a graceful tree, then the corona $T \odot K_1$ is also a graceful tree. The purpose of the note is to generate graceful trees through corona product. If we wish to generate graceful graphs instead of graceful tree, one can think of replacing the graph K_1 with other graphs. In this direction, we raise a question of how to generate graceful graphs using corona products.

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