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Cite as: AIP Conference Proceedings **2516**, 210039 (2022); https://doi.org/10.1063/5.0108661
Published Online: 30 November 2022

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Co-Secure Set Domination Number of Central Graphs

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Abstract. Let $G = (V, E)$ be a non-isolated graph. A Co-Secure dominating set D is said to be a Co-Secure Set Dominating Set if for each subset $T \subseteq V - D$ there exist a non-empty set S of D such that the induced subgraph $\langle T \cup S \rangle$is connected and it's abbreviated as CSSDS of G. The Co-secure set domination number is denoted as $\gamma_{cs}^{s}(G)$ and is defined as the cardinality of smallest co-secure set dominating set.[3] Throughout this paper, we investigate the $\gamma_{cs}^{s}(G)$ of central graph of G. We determine the $\gamma_{cs}^{s}(G)$ of Central graph of path graph, complete graph, cycle graph, star, complete bipartite and wheel graph explicitly and we obtain the sharp bounds for $\gamma_{cs}^{s}(\, C(G))$.

INTRODUCTION

Here, $G = (V, E)$ be a finite, undirected graph with order $|V| = m$ and $|E| = n$ as its size. For any vertex $v \in G, deg(v)\ or\ d(v)$, the degree of '$v$' is exactly the number of edges connected to that vertex v. An open neighbourhood of $u \in V(G)$ is $N_G(u) = \{v \in V\text{: d}(v, u)\text{=1 }\}$. Then the closed neighbourhood of $u \in V(G)$ is defined as $N_G[u] = N_G(u) \cup \{u\}$.

The P_m, path graph of order m and $C_m$is a cycle graph having m vertices. A complete graph K_m of order m with every vertex of degree $m - 1$. The M_1 and N_1 are a bipartite set of $K_{m,n}$, a complete bipartite graph having cardinality m and n respectively. The $K_{1,m}$, star graph with $m + 1$ vertices. The wheel graph W_m, $m \geq 4$ is a graph obtained by joining the centre vertex to every vertex of the cycle graph C_{m-1}. Consider G as a graph with order 'm' and size 'n' and the central graph of G is a graph attained by exactly dividing every element of $E(G)$ once and connecting the vertices which are not adjacent in G and is mentioned as $C(G)$ with $m + n$ order and size ${}^{m}C_2 + n$. Throughout this paper, the central graph with the vertex set, $V(C(G)) = V(G) \cup V_1$, where $V_1 = \{v_{ij}: v_i v_j \in E(G)\}$ and $E(C(G)) = \{v_i v_{ij}, v_j v_{ij}: v_i v_j \in E(G)\} \cup \{v_i v_j: v_i v_j \notin E(G)\}$ is the edge set of $C(G)$.

"A set $D \subseteq V(G)$ is said to be a set dominating set of G if every set $T \subseteq V - D$ there exist a non-empty set S of D such that the induced subgraph $\langle T \cup S \rangle$ is connected. The cardinality of a smallest set dominating set of G is said to be set domination number and it's denoted as $\gamma_s(G)$" and is abbreviated as a SDS(set dominating set) [7]. A dominating set $D \subseteq V(G)$ is a co-secure dominating set of G if for ever vertex $u' \in D$ there exists a vertex $v' \in V - D$ such that $u'v' \in E(G)$and $(D - \{u'\}) \cup \{v'\}$ is a dominating set. It's abbreviated as CSDS of G. The co-secure domination number $\gamma_{cs}(G)$ is the smallest cardinality of a CSDS of G [1]. A co-secure dominating set of G is a co-secure set dominating set of G if for each set $T \subseteq V - D$ there exist a non-empty set S in D such that the induced subgraph $\langle T \cup S \rangle$ is connected and is abbreviated as a CSSDS of G. The CSSDS with smallest cardinality is the co-secure set domination number $\gamma_{cs}^{s}(G)$ of G [3].

The CSDS was initiated by Arumugam S., Karam Ebadi and Martin Manrique and they studied the γ_{cs} of graphs such as path, cycle, compete graph, complete bipartite graph, wheel graph and they also find the sharp bounds for that parameter [1]. For a graph such as Jahangir graph, Helm graph and Friendship graph, the $\gamma_{cs}(G)$ was investigated by Aleena Joseph and Sangeetha and also obtained the bounds [2]. The co-secure set dominating set was initiate by us and we investigated the $\gamma_{cs}^{s}(G)$ of path graph, wheel graph, cycle graph, complete graph, friendship graph, complete

2nd International Conference on Mathematical Techniques and Applications
AIP Conf. Proc. 2516, 210039-1–210039-6; https://doi.org/10.1063/5.0108661
Published by AIP Publishing. 978-0-7354-4234-4/$30.00

bipartite graph and some sharp bounds for it [3]. A graph G with isolated vertices does not have a CSDS and CSSD-set for G. The undefined terms used in this paper are in [5,6].

MAIN RESULTS

Here, we investigate the $\gamma_{cs}^{s}(G)$ of central graphs of some standard graphs explicitly and also obtain the upper bound for $\gamma_{cs}^{s}(C(G))$.

Theorem 2.1

For a path, P_m with $m \geq 2$, then $\gamma_{cs}^{s}\big(C(P_m)\big) = \begin{cases} 2, & if\ m = 2,3 \\ m-1, & if\ m = 4 \\ m-2, & if\ m \geq 5 \end{cases}$.

Proof

Let P_m be a path: $v_1 v_2 v_3 \ldots\ldots v_m$ of order $m \geq 2$ and $v_i v_j \in E(P_m)$ where $j = i+1, 1 \leq i \leq m-1$. Then $V\big(C(P_m)\big) = V \cup V_1$ where $V = V(G)$ and $V_1 = \{v_{i(i+1)}: 1 \leq i \leq m-1\}$.

- Case 1
  For $m = 2$, we have $\gamma_{cs}^{s}\big(C(P_2)\big) = 2$, since $C(P_2) \cong P_3$.
  For $m = 3$, consider $D = \{v_2, v_3\}\ or\ \{v_{12}, v_{23}\}$ is the only CSSDS of $C(P_3)$. Therefore $\gamma_{cs}^{s}\big(C(P_3)\big) = 2$.
- Case 2
  When $m = 4$, $C(P_4)$ has 7 vertices out of that $\{v_{12}, v_{23}, v_{34}\}$ or $\{v_1, v_2, v_3\}$ or $\{v_2, v_3, v_4\}$ is a CSSDS. Therefore, $\gamma_{cs}^{s}\big(C(P_4)\big) = 3$.
- Case 3
  For $m \geq 5$, $C(P_m)$ have $m + E(P_m)$ vertices. Let $D = \{v_2, v_3, \ldots\ldots, v_{m-1}\}$ be a dominating set of $C(P_m)$ with cardinality $m-2$. By the structural nature of $C(P_m)$, the degree of every vertex $v_i \in D$ is $m-1$. So, every vertex $v_i \in D$ can be replaced by the vertex $v_{ij} \in V - D$ such that $v_i v_{ij} \in E\big(C(P_m)\big)$, then the set $(D - \{v_i\}) \cup \{v_{ij}\}$ will be a dominating set of $C(P_m)$. Therefore, D is a CSDS of $C(P_m)$. Since every vertex in D is of degree $m-1$, then every subset T in V-D we can find a non-void subset S in D so that the induced subgraph of $T \cup S$ is connected and D is a SDS of $C(P_m)$. Therefore, D is a CSSDS of $C(P_m)$.

Now, we have to prove that $|D| = m-2$ is a minimum CSSDS of $C(P_m)$, i.e., there exists no CSSDS of $C(P_m)$ with cardinality less than $m-2$. Assume that $|D| < m-2$, i.e., $|D| = m-3$ is a CSSDS of $C(P_m)$. It is clear from the structural nature of $C(P_m)$, D itself is not a dominating set of $C(P_m)$, a contradiction. Therefore, D with cardinality $m-2$ is the minimum CSSDS of $C(P_m)$. Hence, $\gamma_{cs}^{s}\big(C(P_m)\big) = m-2$.

The set $\{v_2, v_3, v_4, v_5\}$ is a minimum CSSDS of $C(P_6)$ as in Figure 1.

Theorem 2.2

If C_m is a cycle graph of order m, then $\gamma_{cs}^{s}\big(C(C_m)\big) = \begin{cases} 3, & if\ m = 3,4 \\ 4, & if\ m = 5 \\ m-2, & if\ m \geq 6 \end{cases}$.

Proof

Let C_m be a cycle graph of vertices $m \geq 3$ and $v_1 v_m$, $v_i v_j \in E\big(C(C_m)\big)$ for $j = i+1, 1 \leq i \leq m-1$. Then $V\big(C(C_m)\big) = V \cup V_1$, where $V = V(C_m)$ and $V_1 = \{v_{i(i+1)}: 1 \leq i \leq m-1\} \cup \{v_{1m}\}$.

- Case 1
  When $m = 3$, $D = \{v_{12}, v_{23}, v_{31}\}$ is a CSSDS of $C(C_3)$. Hence $\gamma_{cs}^{s}\big(C(C_3)\big) = 3$.
  When $m = 4$, $D = \{v_1, v_2, v_3\}$ is a CSSDS of $C(C_4)$. Hence $\gamma_{cs}^{s}\big(C(C_4)\big) = 3$.
- Case 2

For $m = 5$, $D = \{v_2, v_3, v_4, v_5\}$ is a CSSDS of $C(C_5)$. Hence $\gamma_{cs}^{s}\big(C(C_5)\big) = 4$.

- Case 3
  For $m \geq 6$, $C(C_m)$ have $m + E(C_m)$ vertices. Let D be any dominating set with cardinality $m - 2$. And every vertex v_i in D can be replaced by the vertex v_{ij} in V-D such that $v_i v_{ij} \in E\big(C(C_m)\big)$, then the set $(D - \{v_i\}) \cup \{v_{ij}\}$ will be a dominating set of $C(C_m)$. Therefore, D is a CSDS of $C(C_m)$. The D is made up of m-2 vertices from V and V-D consists of 2 vertices from V and the remaining vertices from V_1. Since every vertex in D is of degree m-1. Then every set $T \subseteq V - D$ we can find a non-empty set S in D so that the induced subgraph of $T \cup S$ is connected. Therefore, it's a CSSDS of $C(C_m)$.

Now we have to show that D is a minimum CSSDS of $C(C_m)$ with m-2 vertices. Assume that $|D| < m - 2$, take $|D| = m - 3$ is a CSSDS of $C(C_m)$. Then D with $m - 3$ vertices is not a dominating set of $C(C_m)$, a contradiction. Thus, D with cardinality m-2 is a minimum CSSDS of $C(C_m)$. Hence $\gamma_{cs}^{s}\big(C(C_m)\big) = m - 2$.

The set $\{v_1, v_2, v_4, v_5\}$ is a minimum CSSDS of $C(C_6)$ as in Figure 2.

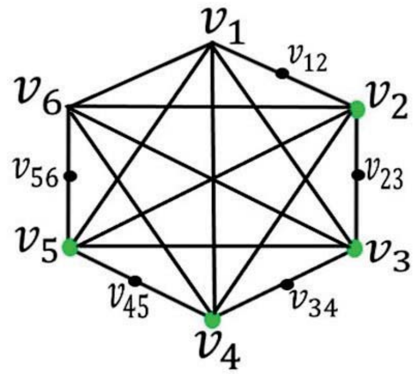

FIGURE 1. A min-CSSDS of $C(P_6)$.

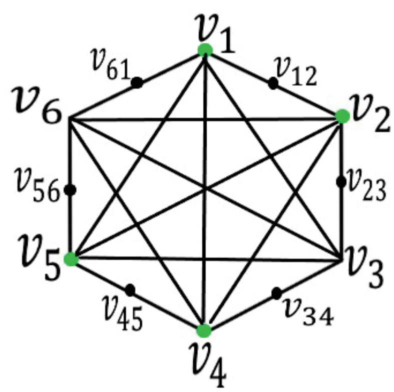

FIGURE 2. A min-CSSDS of $C(C_6)$.

To compare the $\gamma_{cs}^{s}(G)$ of a path and cycle with its co-secure set domination number of central graphs, we use the following two observations which was proved in [3].

Observation [3] 2.3

For a path P_m, $\gamma_{cs}^{s}(P_m) = \begin{cases} m-1, & for\ m = 2,3 \\ m-2, & for\ m = 4,5,6. \\ does\ not\ exist & for\ \ m \geq 7 \end{cases}$

Observation [3] 2.4

If C_m is a cycle graph of m vertices, then $\gamma_{cs}^{s}(C_m) = \begin{cases} m-2, & for\ m = 3,4 \\ m-3, & for\ 5 \leq m \leq 9 \\ does\ not\ exist & for\ \ m \geq 10 \end{cases}$.

But, the $\gamma_{cs}^{s}(C(P_m))$ and $\gamma_{cs}^{s}(C(C_m))$ exists for all m and also $\gamma_{cs}^{s}(P_m) \leq \gamma_{cs}^{s}(C(P_m))$, $\gamma_{cs}^{s}(C_m) \leq \gamma_{cs}^{s}(C(C_m))$.

Theorem 2.5

Let K_m be a complete graph with order m, then $\gamma_{cs}^{s}\big(C(K_m)\big) = \begin{cases} 2, & if\ m = 2 \\ \frac{m(m-1)}{2}, & if\ m \geq 3 \end{cases}$.

Proof

Let K_m be a complete graph: $v_1v_2 \ldots\ldots v_m$ of order $m \geq 2$ and $v_iv_j \in E(K_m)$ if and only if $i < j\ and\ 1 \leq i \leq m, 2 \leq j \leq m$. Then $V\big(C(K_m)\big) = V \cup V_1$ where $V = V(K_m)$ and $V_1 = \{v_{ij} :\ i < j, 1 \leq i \leq m, 2 \leq j \leq m\}$.

Consider $m = 2$, we have $\gamma_{cs}^{s}\big(C(K_2)\big) = 2$. Since $C(K_2) \cong C(P_2)$. For $m \geq 3$, $C(K_m)$ have $m + E(K_m)$ vertices. Let $D = V_1$ be a dominating set of $C(K_m)$ and $|D| = |V_1| = m(m-1)/2$. So, every set in D is adjacent to atmost two vertices in V-D. Then, for every vertex v_{ij} in D can be replaced by a vertex v_i or v_j in V-D such that v_iv_{ij} or $v_{ij}v_j \in E\big(C(K_m)\big)$ and $\big(D - \{v_{ij}\}\big) \cup \{v_i\}\ or\ \big(D - \{v_{ij}\}\big) \cup \{v_j\}$ is a dominating set of $C(K_m)$. Therefore, D is a CSDS of $C(K_m)$. Since every vertex in V-D is of degree m-1 and $deg\big(v_{ij}\big) = 2$ for all $v_{ij}\ in\ D$. Thus, for every subset T in V-D we can find a set S in D such that the subgraph $\langle T \cup S \rangle$ is connected and D is a CSSDS of $C(K_m)$.

Now, we have to prove that D is a minimum CSSDS of $C(K_m)$ with $|D| = \frac{m(m-1)}{2}$. Assume that $|D| < \frac{m(m-1)}{2}$ is a CSSDS of $C(K_m)$. Let us consider $|D| = \frac{m(m-1)}{2} - 1$ be CSSDS of $C(K_m)$. This D itself is not a dominating set of $C(K_m)$, a contradiction. Therefore, D is a minimum CSSDS of $C(K_m)$ with $|D| = \frac{m(m-1)}{2}$. Hence $\gamma_{cs}^{s}\big(C(K_m)\big) = \frac{m(m-1)}{2}$ for $m \geq 3$.

Theorem 2.6

For a star graph $K_{1,m}$, $\gamma_{cs}^{s}\Big(C\big(K_{1,m}\big)\Big) = \begin{cases} 2, & if\ m = 1 \\ m, & if\ m \geq 2 \end{cases}$.

Proof

Let $K_{1,m}$be a star graph with $V\big(K_{1,m}\big) = \{v_1, v_2, \ldots, v_{m+1}\}$ and $v_1v_i \in E\big(K_{1,m}\big)$ for $i = 2\ to\ m + 1$. Then $V\Big(C\big(K_{1,m}\big)\Big) = V \cup V_1$ where $V = V\big(K_{1,m}\big)$ $V_1 = \{v_{1i}: i = 2\ to\ m + 1\}$ and $E\Big(C\big(K_{1,m}\big)\Big) = \{v_1v_{1i}, v_{1i}v_i: for\ 2 \leq i \leq m + 1\} \cup \{v_iv_j: v_iv_j \notin E\big(K_{1,m}\big)\ and\ i < j\ for\ 2 \leq i \leq m, 3 \leq j \leq m + 1\}$.

Since $K_{1,1} \cong P_2$, we have $\gamma_{cs}^{s}\Big(C\big(K_{1,1}\big)\Big) = 2$. Let $D = V_1$ be a dominating set of $C\big(K_{1,m}\big)$ with $|D| = m$. Since every vertex v_{1i} in D is adjacent to atmost two vertices in V-D. So, we can replace every vertex v_{1i} in D by a vertex v_i in V-D such that $v_{1i}v_i \in E\Big(C\big(K_{1,m}\big)\Big)$ then the set $(D - \{v_{1i}\}) \cup \{v_i\}$ is also a dominating set of $C\big(K_{1,m}\big)$. Thus, D is a CSDS of $C\big(K_{1,m}\big)$. Since every vertex in $V - D$ is of degree m and therefore for each set $T \subseteq V - D$ we can find a set S in D such that the $\langle T \cup S \rangle$ is connected. Therefore, D is a CSSDS of $C\big(K_{1,m}\big)$.

We have to prove that D is a minimum CSSDS of $C\big(K_{1,m}\big)$ with $|D| = m$. Assume that $D < m$ is a CSSDS of $C\big(K_{1,m}\big)$, then that D will not be a dominating set, a contradiction. Thus, D with m vertices is a minimum CSSDS of $C\big(K_{1,m}\big)$. Hence, $\gamma_{cs}^{s}\Big(C\big(K_{1,m}\big)\Big) = m$.

Theorem 2.7

For a complete bipartite graph, $K_{m,n}$, the $\gamma_{cs}^{s}\left(C(K_{m,n})\right) = m + n - 1, m \leq n$.

Proof

Let $M = \{v_1, v_2, \ldots., v_m\}$ and $N = \{v_{m+1}, v_{m+2}, \ldots., v_{m+n}\}$ be the partition set of $K_{m,n}$ with order $m + n$ and $v_i v_j \in E(K_{m,n})$ for $1 \leq i \leq m, m + 1 \leq j \leq m + n$. Then $V\left(C(K_{m,n})\right) = V(K_{m,n}) \cup V_1$, where $V_1 = \{v_{ij}, 1 \leq i \leq m, m + 1 \leq j \leq m + n\}$.

If $m = 1$, the $K_{m,n}$ is a star graph and the result as in theorem 2.6. When $m \geq 2$, consider $D = \{v_1, v_2, \ldots., v_{m+n-1}\}$ with $|D| = m + n - 1$. Since every vertex v_i, for $i = 1,2, \ldots m + n - 1$ in D is of degree m+n-1 and $V - D = \{v_{m+n}\} \cup V_1$, then $deg(v_{m+n}) = m + n - 1$ and every v_{ij} in V_1 is of degree 2. Thus, every vertex v_i in D can be replaced by v_{ij} in V-D such that $v_i v_{ij} \in E\left(C(K_{m,n})\right)$ and the new set $(D - \{v_i\}) \cup \{v_{ij}\}$ will act as dominating set for $C(K_{m,n})$. Then, D is a CSDS of $C(K_{m,n})$. For each set T in V-D we can find a subset S in D then the subgraph induced by $\langle T \cup S \rangle$ is connected. Therefore, D is a CSSDS.

To prove that D with cardinality $m + n - 1$ is a minimum CSSDS of $C(K_{m,n})$. suppose that $|D| = m + n - 2$ is a CSSDS of $C(K_{m,n})$. Then that D itself is not a dominating set of $C(K_{m,n})$, a contradiction. Therefore, D is a minimum CSSDS of $C(K_{m,n})$ with cardinality m+n-1. Hence, $\gamma_{cs}^{s}\left(C(K_{1,m})\right) = m + n - 1$.

Theorem 2.8

If W_m a wheel graph, then $\gamma_{cs}^{s}(C(W_m)) = 2m - 2$, for $m \geq 4$.

Proof

Let W_m be a wheel graph of order m and $v_i v_j, v_i v_m \in E(W_m)$ for $j = i + 1$ *and* $1 \leq i \leq m - 1$. Then $V(C(W_m)) = V(W_m) \cup V_1$, where $V_1 = \{v_{im}, v_{ij}: for\ j = i + 1, 1 \leq i \leq m - 1\}$ or $\{v_{im}, v_i\}$ with $|D| = 2m - 2$.

Consider $D = \{v_{ij}, v_{im}\}$ or $\{v_{im}, v_i\}\ for\ j = i + 1, 1 \leq i \leq m - 1$ with $|D| = 2m - 2$ as a dominating set of $C(W_m)$. Every vertex in D is adjacent to atmost 2 vertices of V-D or ($v_i \in D$ is of degree m-1 and v_{im} in D is of degree 2). Then, every vertex $v_{im}\ or\ v_{ij}$ in D can be replaced by v_i in V-D such that $v_{im} v_i\ or\ v_{ij} v_i \in E(C(W_m))$, then the set $(D - \{v_{ij}\}) \cup \{v_i\}$ or $(D - \{v_{im}\}) \cup \{v_m\}$ will act as a dominating set for $C(W_m)$. Thus, D is a CSDS of $C(W_m)$. And for every subset T in V-D we can find a subset S in D then the subset $\langle T \cup S \rangle$ is connected. Therefore, D is a CSSDS of $C(W_m)$.

To prove that D with cardinality $2m - 2$ is a minimum CSSDS of $C(W_m)$. Suppose that $|D| = 2m - 3$, then D itself is not a dominating set of $C(W_m)$. Thus, D with $2m - 2$ vertices is a minimum CSSDS of $C(W_m)$. Hence, $\gamma_{cs}^{s}(C(W_m)) = 2m - 2$.

Theorem 2.9

For any graph G of order $m \geq 3$ with non-isolated vertex, then $2 \leq \gamma_{cs}^{s}(C(G)) \leq \frac{m(m-1)}{2}$. The upper bound is tight by the theorem 2.5.

CONCLUSION

In this paper, we have determined the $\gamma_{cs}^{s}(C(G))$ of path, cycle, complete, star, complete bipartite and wheel graph and we obtained the upper bound for the co-secure set domination number of central graphs. Further we can, Investigate the effect on $\gamma_{cs}^{s}(C(G))$ by removing or adding a vertex to a $C(G)$.

REFERENCES

1. S. Arumugam, Karam Ebadi and Martin Manrique, *Utilitas Mathematica* **94** (2014), pp. 167–182.
2. Aleena Joseph, V. Sangeetha, *International Journal of Mathematics Trends and Technology (IJMTT)*, **5** (2018), pp. 158-164.
3. D. Bhuvaneswari, S. Meenakshi, *International Journal of Recent Technology and Engineering (IJRTE)*, **8** (2019) no. 4S5, pp. 66-68.
4. E. J. Cockayne, P. J. P. Grobler, et.al, *Util. Math.*, **67** (2005), pp.19-32.
5. T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, (Marcel Dekker, Inc. New York, 1998).
6. T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Domination in Graphs: Advanced Topics*, (Marcel Dekker, Inc. New York, 1998).
7. E. Sampath Kumar, and PusphaLatha, *J. Graph theory*, **18** (1994), no. 5, pp. 489–495.