

RAMSEY NUMBERS IN SIERPINSKI TRIANGLE

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Abstract: Sierpinski triangle or Sierpinski gasket S_n is considered as dynamical system. In this paper Ramsey numbers, cycle structure and pathway are discussed in Sierpinski gasket S_n . The connection of Ramsey numbers, cycle structure and pathway leads to Sierpinski graph $S_{n,GR}$.

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1. Introduction

A fractal is generally “a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole, a property called self-similarity”. The term fractal coined by Benoit Mandelbrot in 1975 and was derived from the Latin fractals meaning “broken” or “fractured”. A mathematical fractal is based on recursion. Self-similarity can be quantified as a relative measure of the number of basic building blocks that form a pattern and this measure is defined as Fractal Dimension. The fractal dimension is one of the most commonly used measures to character-

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ize the complexity of a system. There are three types of self-similarity found in fractals, namely, Exact self-similarity, Quasi-self-similarity, Statistical self-similarity. Examples of fractals are Cantor Sets, Sierpinski Triangle, Menger sponge, Dragon Curve, Space-filling Curve, Julia set and Mandelbrot Set [4]. For triangular area, with each iteration, the side of the inside triangle reduces by a factor of 2. The number of these little triangles, on the other hand increases not by 4 but by factor of 3. The dimension of self-similar object is then $(\log 3 / \log 2) = 1.58$ approximately. In this paper in Section 2, we give the regress definitions. In Section 3, the main theorem of the article is proven by using Ramsey numbers.

2. Preliminaries

Definition 2.1 (Sierpinski Graph). The Sierpinski graph $S_{n,GR}$ of order ‘ n ’ is the graph obtained from the connectivity of the Sierpinski graph $S_{n,GR}$.

Definition 2.2 (Ramsey Numbers). • A clique of size ‘ t ’ is a set of ‘ t ’ vertices such that all pairs among them are edges. An independent set of size ‘ s ’ is a set of ‘ s ’ vertices such that there is no edge between them.

- The Ramsey number $R(s, t)$ is the minimum number ‘ n ’ such that any graph on ‘ n ’ vertices contains either an independent set of size ‘ s ’ or a clique of size ‘ t ’.
- The Ramsey number $R_k(s_1, s_2, \dots, s_k)$ is the minimum number ‘ n ’ such that any colouring of the edges of K_n , with ‘ k ’ colours contains a clique of size s_i in colour ‘ i ’ for some ‘ i ’.

Definition 2.3 (Ramsey Numbers for Bipartite). Let, for B_1 and B_2 their bipartite Ramsey numbers is defined as

$$br(B_1, B_2) = \min\{N : K_{N,N} \rightarrow (B_1, B_2)\}$$

The size bipartite Ramsey number $\widehat{br}(B_1, B_2)$ as

$$\widehat{br}(B_1, B_2) = \min\{e(B) : B \text{ is bipartite and } B \rightarrow (B_1, B_2)\}$$

Definition 2.4 (A Bipartite Graph). A complete bipartite graph $G := (V_1 + V_2, E)$ is a bipartite graph such that for any two vertices, $v_1 \in V_1$ and $v_2 \in V_2$, v_1v_2 is an edge in G . The complete bipartite graph with partitions of $|V_1| = m$, and $|V_2| = n$ is denoted $K_{m,n}$.

Definition 2.5 (Path Component). Let S be a subset of the complex plane and let $p \in S$. The path component of S containing p is the set $\{q \in S / p \text{ and } q \text{ are connected by a path in } S\}$.

Definition 2.6 (Monochromatic). The monochromatic triangle problem is an algorithm problem on graphs, in which the goal is to partition the edges of a given graph into two triangle-free sub graphs.

Definition 2.7 (Ramsey's Theorem). Let $m_1, m_2, \dots, m_k \geq 2$ be integers. Then there exists a least positive integer $r(m_1, \dots, m_k)$ with the property that if $p \geq r(m_1, \dots, m_k)$, and if the edges of K_p are partitioned into k classes E_1, E_2, \dots, E_k , then for some i , there is a complete subgraph K_{m_i} all of whose edges are in the class E_i .

Definition 2.8 (Bounded). A set S of real numbers is called bounded from above if there is a real number K such that $K \leq s$ for all s in S . The number K is called an upper bound of S . The terms bounded from below and lower bounds are similarly defined. A set S is bounded if it has upper and lower bounds. Therefore, a set of real numbers is bounded if it is contained in a finite interval.

Definition 2.9 (Complete Subgraph). A clique, C , is an undirected graph $G = (V, E)$ is a subset of the vertices, $C \subseteq V$, such that every two distinct vertices are adjacent. This is equivalent to the condition that the induced sub graph of G induced by C is a complete graph.

Definition 2.10 (Connected Graph). Two points u and v of a graph G are said to be connected if there exists a $u - v$ path in G .

A graph G is said to be connected if every pair of its points are connected.

Definition 2.11 (Cycle). A walk is closed if it has positive length and its origin and terminus are the same.

A closed trail whose origin and internal vertices are distinct is a cycle. Just as with paths we sometimes use the term 'cycle' to denote a graph corresponding to a cycle.

A cycle of length k is called a k -cycle; a k -cycle is odd or even according as k is odd or even.

A 3-cycle is often called a triangle.

3. Methods

In this section, Ramsey numbers is proposed to Sierpinski triangle based on cycle, pathway and connectivity.

Theorem 3.1. S_n has $\frac{3}{2}(3^{n-1} + 1)$ vertices of 3^n edges.

Proof. By the definition 2.3 and 2.4

To construct S_{n+1} from S_n we add one downward facing triangle in each of the 3^{n-1} upward facing triangle of S_n . Thus we add 3^n points. In other words

$$\begin{aligned} |V_{n+1}| &= |V_n| + 3^n \\ &= |V_1| + \sum_{i=1}^n 3^i \\ &= \frac{3}{2}(3^n + 1) \end{aligned}$$

as asserted. The number of edges in S_n may now be easily determined using the fact that the sum of the vertex degrees equals twice the number of edges [1].

$$\begin{aligned} |E_n| &= \frac{1}{2} \sum_{j=1}^{|V_n|} deg(v_j) \\ &= \frac{1}{2} \cdot (4 \cdot \frac{3}{2}(3^{n-1} - 1) + 2 \cdot 3) \\ &= 3^n \end{aligned}$$

Hence the proof of Figure 1.

Iteration number	Scaling	Ramsey Numbers
I_1	1/2	$\{(a, c), (b, c), (a, b)\}$
I_2	1/4	$(a, (a, c)), (a, (a, b)),$ $((a, c), (a, b)),$ $((a, b), b), (b, (b, c)),$ $((a, b), (b, c)),$ $((a, c), (b, c)), ((a, c), c),$ $(c, (b, c))$
\vdots	\vdots	\vdots
I_n	$(1/2)^n$	'n' numbers

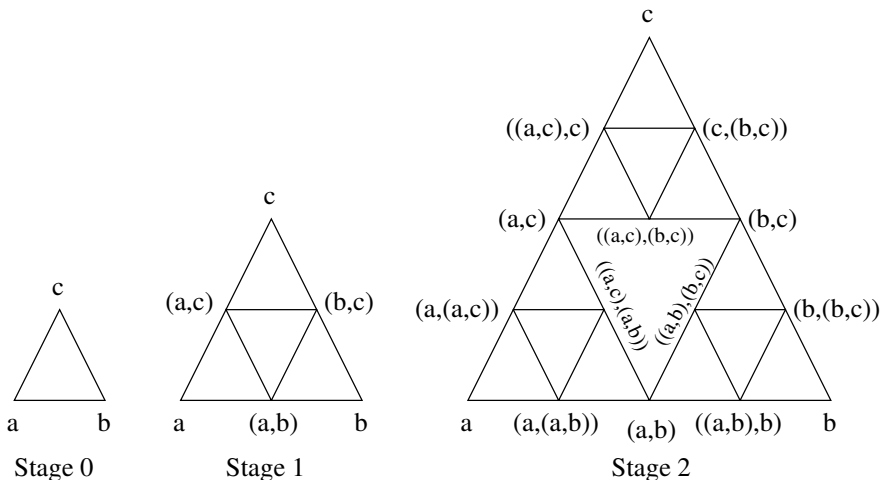


Figure 1: Iterations of Sierpinski triangle

Theorem 3.2. For any graph $S_{n,GR}$ with 6 points or $\overline{S}_{n,GR}$ contains a triangle.

Proof. Let v be a point of $S_{n,GR}$.

Since $S_{n,GR}$ contains 5 points other than that v , v must be either adjacent to three points in G or non-adjacent to three points in G . Hence v must be adjacent to three points either in $S_{n,GR}$ or in $\overline{S}_{n,GR}$.

Without loss of generality, [3] let us assume that v is adjacent to three points u_1, u_2, u_3 in $S_{n,GR}$. If two of these three points are adjacent, $S_{n,GR}$ contains a triangle in $\overline{S}_{n,GR}$. Hence $S_{n,GR}$ or $\overline{S}_{n,GR}$ contains a triangle.

‘6’ is the smallest positive integer such that any graph $S_{n,GR}$ points contains Sierpinski triangle.

Theorem 3.3. A graph $S_{n,GR}$ is connected if and only if for any partition of V into subsets V_1 and V_2 there is a line of $S_{n,GR}$ joining a point of V_1 to a point of V_2 .

Proof. By the definition 2.10., Suppose $S_{n,GR}$ is connected.

Let $V = V_1 \cup V_2$ be a partition of V into two subsets.

Let $u \in V_1$ and $v \in V_2$. Since $S_{n,GR}$ is connected, there exists a $u - v$ path in $S_{n,GR}$, say,

$$u = v_0, v_1, v_2, \dots, v_n = v$$

Let i be the positive integer such that $v_i \in V_2$. (Such an i exists since $v_n = v \in V_2$)

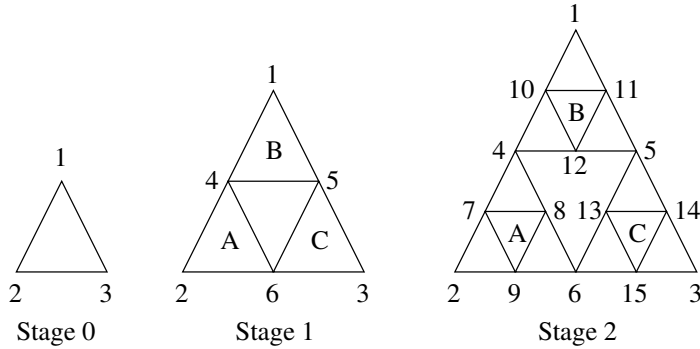


Figure 2: Sierpinski graph

Then $v_{i-1} \in V_1$ and v_{i-1}, v_i are adjacent. Thus there is a line joining $v_{i-1} \in V_1$ and $v_i \in V_2$.

To prove the converse, Suppose $S_{n,GR}$ is not connected. Then $S_{n,GR}$ contains at least two components.

Let V_1 denote the set of all vertices of one component and V_2 the remaining vertices of $S_{n,GR}$. Clearly $V = V_1 \cup V_2$ is a partition of V and there is no line joining any point of V_1 to any point of V_2 .

Hence the theorem.

Theorem 3.4. *A graph $S_{n,GR}$ with at least two points is bipartite if and only if all its cycles are of even length.*

Proof. By the definition 2.4 and 2.11;

Suppose $S_{n,GR}$ is Sierpinski Graph (Fractal triangle). [2] Then V can be partitioned into two subsets V_1 and V_2 such that every line joins a points of V_1 to a point of V_2 . Now consider any cycle $v_0, v_1, v_2, \dots, v_n = v_0$ of length n .

Suppose $v_0 \in V_1$, then $v_2, v_4, v_6, \dots \in V_1$ and $v_1, v_3, v_5, \dots \in V_2$, further $v_n = v_0 \in V_1$ and hence n is even.

Conversely, suppose all cycles in $S_{n,GR}$ are of even length. We may assume without loss of generality that $S_{n,GR}$ is connected.

Let $v_1 \in V$

Define $V_1 = \{v \in V/d(v, v_1) \text{ is even}\}$

$V_2 = \{v \in V/d(v, v_1) \text{ is odd}\}$

Clearly, $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = V$.

We claim that,

every line of $S_{n,GR}$ joins a point of V_1 to a point of V_2 .

Suppose two points $u, v \in V_1$ are adjacent.

Let P be a shortest $v_1 - u$ path of length m and let Q be a shortest $v_1 - v$ path of length n . Since $u, v \in V_1$ both m and n are even. Now, let u be the last point common to P and Q . Then the $v_1 - u_1$ path along P and the $v_1 - u_1$ path along Q are both shortest paths and hence have the same length, say i .

Now the $u_1 - u$ path along P , the line uv followed by the $v - u_1$ path along Q form a cycle of length $(m - i) + 1 + (n - i) = m + n - 2i + 1$ which is odd and this is a contradiction.

Thus no two points of V_1 are adjacent, hence $S_{n,GR}$ is bipartite.

In all iterations based on scaling have even length.

Theorem 3.5. *If $\delta \geq K$, then $S_{n,GR}$ has path of length K .*

Proof. $\delta(S_{n,GR}) = \min\{\text{deg } v/v \in V(S_{n,GR})\}$; K - edges

By the definition 2.5;

Let v_1 be an arbitrary point, choose v_2 adjacent to v_1 .

Since $\delta \geq K$, there exists at least $K - 1$ vertices other than v_1 which are adjacent to v_2 .

Choose $v_3 \neq v_1$, such that v_3 is adjacent to v_2 .

In general, [5] having chosen $v_1, v_2, v_3, \dots, v_i$ where $1 < i \leq \delta$ there exists a point $v_{i+1} \neq v_1, v_2, v_3, \dots, v_i$ such that v_{i+1} is adjacent to v_i . This process gives a path of length K in $S_{n,GR}$.

Theorem 3.6. *For any two natural numbers, s and t , there exists a natural number, $R(s, t) = n$, such that any 2-coloured complete graph of order at least n , coloured red and blue, must contain a monochromatic red K_s or blue K_t .*

Proof. By the definition 2.7, 2.8, 2.9;

To prove that $R(s, t)$ exists by proving it is bounded.

Proof by induction method, [2] assume that $R(s - 1, t)$ and $R(s, t - 1)$ exists.

“The Ramsey numbers, $R(s, t)$ is the order of the smallest complete graph which, when 2-coloured must contain a red K_s or a blue K_t ”

$R(s, t) = R(t, s)$ since the color of each edge can be swapped. Two simple results are $R(s, 1) = 1$ and $R(s, 2) = s$. $R(s, 1) = 1$ is trivial since K_1 has no edges and so no edges to color, thus any colouring of K_1 will always contain a blue K_1 . $R(s, 2) = s$ is also a simple result; if all the edges of K_s are coloured red, it will contain a red K_s , however if one edge is coloured blue it will contain a blue K_2 . The edges of any graph of order less than s could all be coloured red in which case the graph would contain neither a red K_s or a blue K_2 .

$R(s, 2) = R(2, s) = s$ and

$R(s, 1) = R(1, s) = 1$ are trivial results by earlier.

Claim:

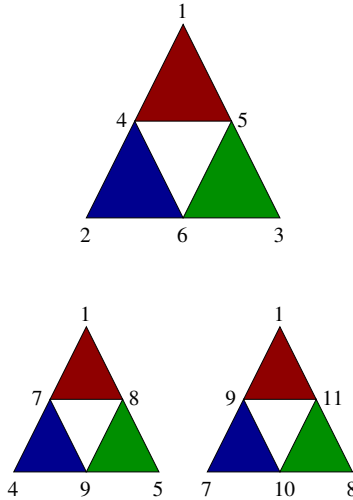


Figure 3: Coloured graph for Sierpinski triangle

$$R(s, t) \leq R(s - 1, t) + R(s, t - 1)$$

A complete graph of 2-colouring with $n = R(s - 1, t) + R(s, t - 1)$ vertices.

Let x be the two sets R_x and B_x , are the set of vertices adjacent to x , and every edge connecting a vertex in R_x to x as red, and B_x to x as blue.

Since K_n is a complete graph

$$B_x = [n](R_x \cup \{x\}) \text{ and so } |R_x| + |B_x| = n - 1$$

If $|R_x| < R(s - 1, t)$ and $|B_x| < R(s - 1, t)$

Since $n = R(s - 1, t) + R(s, t - 1)$ and $|R_x| + |B_x| \leq n - 2$ a contradiction.

So, $|B_x| \geq R(s, t - 1)$ or $|R_x| \geq R(s - 1, t)$.

If $|B_x| \geq R(s, t - 1)$ and B_x induces a red K_s ,

If B_x induces a blue K_{t-1} then K_n must contain a blue K_t .

Since $B_x \cup \{x\}$ must induce a blue K_t .

Edge xt is blue for all $t \in B_x$, by definition of B_x .

Therefore $B_x \cup \{x\}$ must induce a blue K_t if B_x contains a blue K_{t-1} .

R_x is completely symmetric, if R_x induces a blue K_t and if R_x induces a red K_{s-1} then K_n must contain a red K_s .

Since $R_x \cup \{x\}$ must induce a red K_s .

A 2-coloured complete graph of order $R(s - 1, t) + R(s, t - 1)$ must contain a red K_s or a blue K_t . $R(s, t) \leq R(s - 1, t) + R(s, t - 1)$

Hence the theorem is proved.

This theorem is applicable to Sierpinski triangle S_n .

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