A Note On Advanced Normalization Algorithms And Their Applications

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Abstract—**The max-plus semiring is a semiring where the addition operation is replaced with maximum operation and the multiplication operation is replaced with the usual addition. In this paper we suggest 'Advanced normalization method' and its algorithms to solve the max linear system and its applications are discussed. We use these algorithms to determine the existence, uniqueness and the principal solution of a max linear system. An important aspect of the advanced normalization method is finding the free variables and the leading variables of any solvable max linear system. We have also found the generalized principal solution of the max linear system with some special matrices such as row-wise arithmetic matrix, column-wise arithmetic matrix, ϕ- diagonal matrix, $J_{c\times d}(l)$ matrix, arithmetic circulant matrix. The generalized principal solution of the max linear system had been found in both cases, $T\bigodot x=u$ with $u\neq 0$ and $T\bigodot x=0$. We have given some important results on the principal solution of the max linear system.**

Index Terms—**Max linear systems, Principal solution, Degrees of freedom.**

I. Introduction

TROPICAL semiring [1] is a semiring, in which the addition operation is replaced with maximum or minimum operation and multiplication operation is replaced by usual addition [2]. If we use maximum then the semiring is called max-plus semiring. We call the tropical semiring as min-plus then minimum is stands for addition. Tropical semiring is generally used in optimization and computer science for decades, its connection to algebraic geometry was found only at the beginning of this century. Tropical geometry was introduced by Imre Simon, from Brazil [3], [4], [5], [6], [7]. He is a mathematician and a computer scientist. Simon's main work was the reformulation of the finite power property as a Burnside problem utilizing tropical semiring in automata with multiplicities [8]. French mathematicians coined the word 'tropical' to honor Simon's contributions in extending min-plus algebra to optimization theory.

In tropical geometry, max-plus semirings play an poportant role. There are two types of tropical semirings, one is the min-plus semiring where the addition operation is replaced with minimum operation and multiplication operation is replaced with usual addition, the other is the max-plus semiring where the addition operation is replaced with maximum operation and multiplication operation is replaced with the usual addition [9], [10], [11], [12]. The max-plus semiring is also called the maximum tropical semiring. The min-plus semiring is also called as minimum tropical semiring [1].

Max-plus semiring and min-plus semiring are isomorphic to each other. Max-plus semirings are idempotent semirings [9], [13]. In this paper, we have chosen the max-plus semiring to work with. The max-plus semiring $(\mathbb{W}\cup -\infty, \bigoplus, \bigodot)$ was introduced by Simons [3]. The max-plus addition is denoted by $\bigoplus$ and the max-plus product is denoted by $\bigodot$. The main motive of using max-plus algebra is to make calculations faster and its applications in any of the following fields[14], [15].

Max-plus semiring has important applications in computer science, linear algebra, number theory [16], automata theory, language theory, control theory, operation research [17], [18]. In cryptography max-plus semirings were used to frame a new key exchange protocols [19], [20], [21], [22], [23], [24], [28], [26]and algebraic methods were used to attack a key exchange protocol [27], [28]. Also we can use max-plus concepts in graph theory [29], topological transformation groups[30]. Max-plus semirings have significant applications in enumerative geometry [31], [32], classical geometry [33], intersection theory, representation theory, algebraic statistics, mathematical biology [34], [35]. In cryptography, the choleskey decomposition of matrices over the symmetrized max-plus algebra can be used to generate novel key exachange protocols[36]. Solving the linear system is one of the best aspects of linear algebra. Solving it over the max-plus semirings is more tricky [37], [38]. Especially solving the key exchange protocols in cryptography [39], [40]. We have analysed the normalization method, pseudo inverse method, grigoriev's method for solving the max linear system[41], [42], [43]. The complexity of solving max linear system was briefly discussed [44].

In Section 2, the prerequisites of this paper is given. In Section 3, the advanced normalization method is introduced with some examples and some results are provided. Section 4 includes algorithms to check the existence, uniqueness and the principal solution of a solvable max linear system by using the advanced normalization method. Section 5 begins with basic definitions and generalized principal solution of the max linear system where $T\bigodot x=u$ in both $u=0$ and $u\neq 0$. In Section 6, we introduce a method to find the free and leading variables of any max linear system using the advanced normalization method.

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II. Preliminaries

The first axiomatic definition of a ring was given by Abragam Fraenkel in 1914 [45]. A ring is a non-empty set R equipped with two binary operations '+' and '·' called addition and multiplication such that, (i)

1) $(R, +)$ is an abelian group.
2) $(R, \cdot)$ is a semigroup.
3) Right and left distributive law holds.

In 1934, the notion of semiring was first introduced by Vandiver [46] as a universal algebra with two associative binary operations '+' and '·' which are connected by the ring like distributive laws.

Definition 2.1. *[47] A non-empty set containing two binary operations, such as addition and multiplication is called a semiring S if that meets the following axioms:*

1) *S is an abelian monoid under the operation addition with '0' as a unique identity element.*
2) *S is a monoid under the operation multiplication that has a unique identity element which is denoted as '1'.*
3) *$u(v+w)$ is equal to $uv+uw$ and $(v+w)u$ is equal to $vu+wu$, $\forall\ u, v, w \in S$.*
4) *Both $u.0$ and $0.u$ are equal to 0, $\forall\ u \in S$.*
5) *The identities under the two operations should not be the same element.*

Definition 2.2. *[13] A semiring S which satisfies the condition that $a+a$ is equal to a, $\forall\ a \in S$ is called an idempotent semiring.*

Definition 2.3. *[9], [13] A semiring that satisfies the condition $a+b$ is equal to $0 \implies a$ and b are equal to 0 is called a zero sum free.*
Max-plus semiring is an idempotent semiring and all idempotent semirings are zero sum free.

Definition 2.4. *[10], [42], [43] The max-plus semiring is the semiring $R = (S \cup (-\infty), \bigoplus, \bigodot)$ where the operations $\bigoplus$ and $\bigodot$ denoted the max-plus addition and max-plus multiplication respectively. Since S is a semiring, it should satisfy the following properties,*

1) $u \bigoplus v = v \bigoplus u$ *,* $\forall\ u, v \in R$.
2) $(u \bigoplus v) \bigoplus w = u \bigoplus (v \bigoplus w)$ *and* $(u \bigodot v) \bigodot w = u \bigodot (v \bigodot w)$ $\forall\ u, v, w \in R$.
3) $u \bigodot (v \bigoplus w) = (u \bigodot v) \bigoplus (u \bigodot w)$ $\forall\ u, v, w \in R$.
4) $\exists\ e \in R$, $\forall\ u \in R$ *such that* $e \bigoplus u = u \bigoplus e = u$ *(Here the additive identity is $-\infty$)*
5) *Additive inverse does not exist.*

A. Matrices over the max-plus semiring

Collection of all matrices [48], [49], [50], [51] over the semiring S with m rows and n columns denoted as $M_{m\times n}(S)$. Every ij^{th} element of $P \in M_{m\times n}(S)$ matrix mentioned as p_{ij} and transpose of the matrix P is denoted as P^T. Let $P = (p_{ij}) \in M_{m\times n}(S)$, $Q = (q_{ij}) \in M_{m\times n}(S)$, $T = (t_{ij}) \in M_{n\times l}(S)$ and $\alpha \in S$. In max-plus algebra addition of two matrices $P \bigoplus Q$ can be calculated by $(max((p_{ij}), (q_{ij})))_{m\times l}$ and multiplication of two matrices P$\bigodot$T is calculated by

$$max((p_{ik}) + (t_{kj}))_{m\times l}$$

$$\alpha \bigodot P = (\alpha + (p_{ij}))_{m\times n}$$

Definition 2.5. *[52], [53] The linear system $T \bigodot x = u$ is said to be a tropical linear system if the entries of the linear system are all from either max-plus semiring $R = (S \cup (-\infty), \bigoplus, \bigodot)$ or min-plus semiring $R' = (S \cup (\infty), \bigoplus, \bigodot)$.*

Definition 2.6. *[54] A matrix $P \in M_{m\times n}(S)$ is called the max-plus matrix if the entries of P matrix are all from the max-plus semiring $R = (S \cup (-\infty), \bigoplus, \bigodot)$.*

Definition 2.7. *[42] Let $S = \mathbb{R}$ be the real number system under the max-plus algebra. Let P and Q be $m \times n$ matrices over the real numbers under the operations of the max-plus semiring, where p_{ij} and q_{ij} are ij^{th} entries of P and Q respectively. $P \le Q \leftrightarrow p_{ij} \le q_{ij}\ \forall\ i, j$.*

Definition 2.8. *[43] In max-plus semiring, a vector $q = (q_1, q_2, ..., q_m)$ is said to be a normal vector or regular vector if $q_j \neq -\infty$, $\forall\ 1 \le j \le m$. In min-plus semiring, $q = (q_1, q_2, ..., q_m)$ is the regular vector if $q_j \neq \infty$, $\forall\ 1 \le j \le m$.*

Definition 2.9. *[42] A solution y^* of the max linear system $T \bigodot y = u$ is called the principal solution of the max linear system if $y_i \le y^*$, where y_i's are the solutions of the max linear system.*

Definition 2.10. *[53] Let $Tx = u$ is a solvable system of linear equations. Suppose $T_{j_1}, T_{j_2}, T_{j_3}, \cdots T_{j_k}$ are columns of T which are linearly independent and u is a linear combination of $T_{j_1}, T_{j_2}, T_{j_3}, \cdots T_{j_k}$. Then the corresponding variables $x_{j_1}, x_{j_2}, x_{j_3}, \cdots x_{j_k}$ are called leading variables and all other variables are called free variables. The number of free variables is called as degrees of freedom and it denoted by DOF.*

Definition 2.11. *[43] A matrix $P \in M_{m\times n}(S)$ is called a regular matrix or normal matrix under the max-plus operation if $p_{ij} \neq -\infty$, $\forall\ 1 \le i \le m$, $1 \le j \le n$.*

Definition 2.12. *A matrix $P \in M_{m\times m}(S)$ is called a diagonal matrix if all the non diagonal entries are zero.*

Definition 2.13. *[49] A matrix $Z \in M_{n\times n}(S)$ is said to be a circulant matrix if the entries of Z matrix are only with the values $z_0, z_1, \cdots, z_{n-1} \in S$ and these entries are placed in the following form*

$$\begin{bmatrix} z_0 & z_{m-1} & z_{m-2} & \cdots & z_1 \\ z_1 & z_0 & z_{m-1} & \cdots & z_2 \\ z_2 & z_1 & z_0 & \cdots & z_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{m-1} & z_{m-2} & z_2 & \cdots & z_0 \end{bmatrix}$$

Let us consider some tropical semirings **G**=$(\mathbb{R} \cup (-\infty), \bigoplus, \bigodot)$ where $\mathbb{R}$ is a collection of all numbers which are real, **H**=$(\mathbb{Z} \cup (-\infty), \bigoplus, \bigodot)$ where $\mathbb{Z}$ is a collection of all integers, **I**=$(\mathbb{W} \cup (-\infty), \bigoplus, \bigodot)$ where $\mathbb{W}$ is a collection of all whole numbers.

Definition 2.14. *A matrix $T \in M_{c\times d}(\mathbf{G})$ is said to be a row-wise arithmetic matrix with constant term $a \in \mathbb{R}$ and difference $k \in \mathbb{R}$ if*

$$\begin{bmatrix} a+k & a+2k & \cdots & a+dk \\ a+(d+1)k & a+(d+2)k & \cdots & a+2dk \\ \vdots & \vdots & \vdots & \vdots \\ a+((c-1)d+1)k & a+((c-1)d+2)k & \cdots & a+cdk \end{bmatrix}$$

Definition 2.15. *A matrix $T \in M_{c\times d}(\mathbf{G})$ is said to be a column-wise arithmetic matrix with constant term $a \in \mathbb{R}$ and difference $k \in \mathbb{R}$ if*

$$\begin{bmatrix} a+k & a+(c+1)k & \cdots & a+((d-1)c+1)k \\ a+2k & a+(c+2)k & \cdots & a+((d-1)c+2)k \\ \vdots & \vdots & \vdots & \vdots \\ a+ck & a+2ck & \cdots & a+cdk \end{bmatrix}$$

Definition 2.16. *A matrix $T \in M_{c\times d}(\mathbf{G})$ is called the $\phi-$diagonal matrix if $t_{ii} = \phi$ for $1 \leq i \leq c$ and all other elements are zero.*

$$\begin{bmatrix} \phi & 0 & 0 & \cdots & 0 \\ 0 & \phi & 0 & \cdots & 0 \\ 0 & 0 & \phi & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \phi \end{bmatrix}.$$

Definition 2.17. *A matrix $T \in M_{c\times d}(\mathbf{G})$ is said to be a $J_{c\times d}(l)$ matrix if all the entries of T matrix are equal to l.*

$$\begin{bmatrix} l & l & l & \cdots & l \\ l & l & l & \cdots & l \\ l & l & l & \cdots & l \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l & l & l & \cdots & l \end{bmatrix}.$$

Definition 2.18. *A matrix $T \in M_{c\times d}(\mathbf{G})$ is said to be a k^{th} row-wise arithmetic matrix if*

$$\begin{bmatrix} k & 2k & \cdots & dk \\ dk+k & dk+2k & \cdots & 2dk \\ 2dk+k & 2dk+2k & \cdots & 3dk \\ \vdots & \vdots & \vdots & \vdots \\ (c-1)dk+k & (c-1)dk+2k & \cdots & cdk \end{bmatrix}.$$

Definition 2.19. *A matrix $T \in M_{c\times d}(\mathbf{G})$ is said to be a k^{th} column-wise arithmetic matrix if*

$$\begin{bmatrix} k & ck+k & 2ck+k & \cdots & (d-1)mk+k \\ 2k & ck+2k & 2ck+2k & \cdots & (d-1)mk+2k \\ 3k & ck+3k & 2ck+3k & \cdots & (d-1)mk+3k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ ck & 2ck & 3ck & \cdots & cdk \end{bmatrix}$$

Definition 2.20. *Let $Z \in M_{c\times c}(\mathbf{F})$ be a circulant matrix and $T \in M_{c\times d}(\mathbf{F})$ is said to be an arithmetic circulant matrix if entries $z_0, z_1, \ldots, z_{n-1}$ are equal to the entries $1, 2, \ldots, c$ respectively*

$$\begin{bmatrix} 1 & c & c-1 & \cdots & 2 \\ 2 & 1 & c & \cdots & 3 \\ 3 & 2 & 1 & \cdots & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c & c-1 & c-2 & \cdots & 1 \end{bmatrix}$$

III. Advanced normalization method

Suppose the given max linear system is $Tx = u$

$$(t_{11} \odot x_1) \oplus (t_{12} \odot x_2) \oplus \cdots \oplus (t_{1n} \odot x_n) = u_1$$

$$(t_{21} \odot x_1) \oplus (t_{22} \odot x_2) \oplus \cdots \oplus (t_{2n} \odot x_n) = u_2$$

$$\vdots$$

$$(t_{m1} \odot x_1) \oplus (t_{m2} \odot x_2) \oplus \cdots \oplus (t_{mn} \odot x_n) = u_m$$

We have to solve the following equations.

$$\max\{t_{11} \odot x_1, t_{12} \odot x_2, \cdots, t_{1n} \odot x_n\} = u_1$$

$$\max\{t_{21} \odot x_1, t_{22} \odot x_2, \cdots, t_{2n} \odot x_n\} = u_2$$

$$\vdots$$

$$\max\{t_{m1} \odot x_1, t_{m2} \odot x_2, \cdots, t_{mn} \odot x_n\} = u_m$$

Advanced normalization method is introduced to solve the max linear system by defining "Advanced normalized (AN_{Tu})" matrix as follows,

$$AN_{Tu} = \begin{bmatrix} u - T_1 & u - T_2 & \cdots & u - T_n \end{bmatrix}$$

After finding AN_{Tu} matrix find the minimum element in each columns and draw a box around that element. If every row has atleast one boxed element, then the max linear system has a solution. If the system has a solution then we may find the principal solution.

$$x^* = \begin{bmatrix} \min\{d_{11}, d_{21}, \ldots, d_{m1}\} \\ \min\{d_{12}, d_{22}, \ldots, d_{m2}\} \\ \min\{d_{13}, d_{23}, \ldots, d_{m3}\} \\ \vdots \\ \min\{d_{1n}, d_{2n}, \ldots, d_{mn}\} \end{bmatrix}$$

since d_{ij} is a ij^{th} entry of AN_{Tu}. By analyzing the method of advanced normalization method we have given the following results to check whether the given system has a solution.

Theorem 3.1. *Let $T \in M_{m\times n}(S)$ and the linear system $T \bigodot x = u$ with max-plus semiring has a solution if every row of AN_{Tu} matrix has at least one boxed element and the converse is also true.*

Proof: *Assume that the system $T \bigodot X = u$ has a solution. suppose some i^{th} row $1 \leq i \leq n$ of the AN_{Tu} matrix does not have any boxed element. We can say that AN_{Tu} does not have a boxed element in that corresponding i^{th} row $\implies x_j < u_i - t_{ij}$ for all $1 \leq j \leq n \implies max(t_{i1} + x_1, t_{i2} + x_2, \cdots, t_{in} + x_n) < u_i$. This contradicts our assumption. Conversely if every row of AN_{Tu} has at least one boxed element $\implies x_j = u_i - t_{ij}$, $\forall\ 1 \leq i \leq m$ and $1 \leq j \leq n \implies max(t_{i1} + x_1, t_{i2} + x_2, ..., t_{in} + x_n) = u_i \implies$ system has a solution.* ■

Theorem 3.2. *Let $T \otimes x = u$ be a max linear system, AN_{Tu} matrix contains at least one row with more than one column minimum element iff the system has infinitely many solutions.*

Proof: Assume the matrix AN_{Tu} contains at least one row with more than one column minimum element. Suppose row i has column minimum elements in columns j_1 and j_2, then $x_{j_1} = u_i - t_{ij_1}$ and $x_{j_2} = u_i - t_{ij_2}$. Since x_{j_1} and x_{j_2} can both independently satisfy the system, there are infinite number of solutions that satisfy the system. Conversely, assume that system has infinitely many solutions. For any given i the equation in the tropical system is $\max\{t_{i1} + x_1, t_{i2} + x_2, \cdots, t_{in} + x_n\} = u_i$. If there are infinitely many solutions, it must be in at least one row, there are multiple values x_j and x_k that satisfy this equation simultaneously. This implies that the row i of AN_{Tu} must contain more than one column minimum element, corresponding to the fact that both x_j and x_k can independently satisfy the equation.

■

Theorem 3.3. *Let $T \otimes x = u$ be a max linear system. Every row of the AN_{Tu} matrix has exactly one column minimum element iff the system has a unique solution.*

Proof: Assume that every row of the matrix AN_{Tu} has exactly one column minimum element. From the definition of the matrix AN_{Tu} each entry is given by $AN_{Tu}(i, j) = u_i - t_{ij}$. If each row of AN_{Tu} has exactly one column minimum element, then for each row i there exists a unique j such that $x_j = u_i - t_{ij}$. This implies $\max\{t_{i1} + x_1, t_{i2} + x_2, \cdots, t_{in} + x_n\} = u_i$ for exactly one x_j for each j. Conversely, assume that $T \otimes x = u$ has unique solution x^*. Suppose that some row i of the matrix AN_{Tu} has more than one column minimum element. This would mean that for some row i, here are at least two distinct indices j_1 and j_2 such that $u_i - t_{ij_1} = u_i - t_{ij_2}$. This implies that both x_{j_1} and x_{j_2} could satisfy the equation$\max\{t_{i1} + x_1, t_{i2} + x_2, \cdots, t_{in} + x_n\} = u_i$ Thus, the system would have multiple possible solutions, contradicting the assumption that the system has a unique solution. Therefore, every row of AN_{Tu} must have exactly one column minimum element in order to ensure the uniqueness of the solution.

■

IV. Algorithms for solving max linear systems

We introduced three algorithms for solving max linear systems by using the advanced normalization method. AN_{Tu} matrix also called as $D-$ matrix.

Algorithm 1: To check whether the max linear system has a solution

Input : $T_{m\times n}$ matrix and u vector
Output: Whether the max linear system has a solution or not

Advanced Normalization = [ ]
for $i \leftarrow 1$ **to** m *and* $j \leftarrow 1$ **to** n **do**
  Define D-matrix with $AN_j = u - T_j$
end for
for $i \leftarrow 1$ **to** m *and* $j \leftarrow 1$ **to** n **do**
  if $d_{ij} = min(u - T_j)$ **then**
    draw a box around the d_{ij}
  end if
end for
for $i \leftarrow 1$ **to** m *and* $j \leftarrow 1$ **to** n **do**
  if *Every row of D-matrix contains atleast one boxed element* **then**
    system has a solution
  else
    system has no solution
  end if
end for
return $AdvancedNormalization$

Example 4.1. *Consider the max linear system*

$$\begin{bmatrix} 4 & 3 & 7 & 12 & 1 \\ 13 & 15 & 5 & 2 & 5 \\ 9 & 12 & 11 & 7 & 8 \\ 6 & 2 & 4 & 5 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 9 \\ 12 \end{bmatrix}$$

where

$$AN_{Tu} = \begin{bmatrix} -1 & 0 & \boxed{-4} & \boxed{-9} & 2 \\ \boxed{-6} & \boxed{-8} & 2 & 5 & 2 \\ 0 & -3 & -2 & 2 & 1 \\ 6 & 10 & 8 & 7 & \boxed{-5} \end{bmatrix}$$

Third row of the D- matrix has no boxed element $\implies$ system has no solution. Suppose we have 10×10 matrix there is a difficulty to solve such max linear systems. We developed the Algorithm 1 to check whether the linear system has a solution or not.

Example 4.2. *Input :*

$$\begin{bmatrix} 6 & 1 & 2 & 4 & 3 & 2 & 3 & 5 \\ 2 & 3 & 4 & 5 & 1 & 4 & 5 & 4 \\ 2 & 7 & 8 & 9 & 6 & 4 & 5 & 3 \\ 2 & 3 & 4 & 5 & 3 & 2 & 7 & 5 \\ 12 & 12 & 11 & 13 & 10 & 2 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 & 7 & 12 & 20 \\ 2 & 3 & 4 & 5 & 6 & 6 & 7 & 2 \\ 15 & 12 & 13 & 11 & 13 & 14 & 15 & 13 \\ 2 & 3 & 6 & 9 & 7 & 5 & 3 & 2 \\ 12 & 13 & 12 & 4 & 2 & 8 & 6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 23 \\ 16 \\ 17 \\ 18 \\ 19 \\ 2 \end{bmatrix}$$

We have a difficulty to solve the above problem manually. To skip the complexity we used our Algorithm 1 and we executed the Algorithm 1 in scilab. "D-Matrix" obtained as,

$$\begin{bmatrix} -4 & 1 & 0 & -2 & -1 & 0 & -1 & -3 \\ 10 & 9 & 8 & 7 & 11 & 8 & 7 & 8 \\ 21 & 16 & 15 & 14 & 17 & 19 & 18 & 20 \\ 14 & 13 & 12 & 11 & 13 & 14 & 9 & 11 \\ 5 & 5 & 6 & 4 & 7 & 15 & 13 & 12 \\ 16 & 15 & 14 & 13 & 12 & 11 & 6 & -2 \\ 17 & 16 & 15 & 14 & 13 & 13 & 12 & 17 \\ \underline{-13} & \underline{-10} & \underline{-11} & \underline{-9} & \underline{-11} & \underline{-12} & \underline{-13} & \underline{-11} \\ 10 & 9 & 6 & 3 & 5 & 7 & 9 & 10 \\ -1 & -2 & -1 & 7 & 9 & 3 & 5 & 8 \end{bmatrix}$$

Output: "System has no solution"

Example 4.3. *Consider the max linear system*

$$\begin{bmatrix} 15 & 20 & 12 & 14 & 11 \\ 14 & 19 & 15 & 15 & 15 \\ 12 & 12 & 17 & 12 & 12 \\ 11 & 10 & 12 & 11 & 12 \end{bmatrix} \bigodot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \\ 12 \\ 12 \end{bmatrix}$$

where

$$AN_{Tu} = \begin{bmatrix} \boxed{-4} & \boxed{-9} & -1 & \boxed{-3} & \boxed{0} \\ 1 & -4 & 0 & 0 & \boxed{0} \\ 0 & 0 & \boxed{-5} & 0 & \boxed{0} \\ 1 & 2 & 0 & 1 & \boxed{0} \end{bmatrix}$$

by an Algorithm 1 we decided that the given system has a solution. By Theorem 3.2 we obtained that the system has many solutions. If the system has many solutions then we *can conclude all other solutions by fixing one element of x^* and varing all other elements of x^*. In this example principal solution is*

$$x^* = \begin{bmatrix} -4 \\ -9 \\ -5 \\ -3 \\ 0 \end{bmatrix}$$

for all other solutions we should vary the entries as follows,

$$x = \begin{bmatrix} -4 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}; a_1 \leq -9, a_2 \leq -5, a_3 \leq -3, a_4 \leq 0;$$

$$x = \begin{bmatrix} b_1 \\ -9 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}; b_1 \leq -4, b_2 \leq -5, b_3 \leq -3, b_4 \leq 0;$$

$$x = \begin{bmatrix} c_1 \\ c_2 \\ -5 \\ c_3 \\ c_4 \end{bmatrix}; c_1 \leq -4, c_2 \leq -9, c_3 \leq -3, c_4 \leq 0;$$

$$x = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ -3 \\ d_4 \end{bmatrix}; d_1 \leq -4, d_2 \leq -9, d_3 \leq -5, d_4 \leq 0;$$

$$x = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ 0 \end{bmatrix}; e_1 \leq -4, e_2 \leq -9, e_3 \leq -5, e_4 \leq -3;$$

In this problem we solved max linear system manually but in the case of the coefficient matrix has higher order then it will be more difficult to check whether the max linear system has a unique solution or many solutions. We developed an Algorithm 2 to avoid this difficulty.

Algorithm 2: To find the principal solution of the max linear system

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Input : $T_{m \times n}$ matrix and u vector
Output: Principal solution of the max linear system

```
Principalsolution = [ ]
for j ← 1 to n do
    if the given system has a solution then
        j^th element of the principal solution can be
        obtain with the formula min(u − T_j), where
        T_j denotes the j^th column of T-matrix.
    end if
end for
return Principal solution
```

V. Results

We begin this section with some basic definitions. In this section by using the method of a advanced normalization we have given the upcoming results for the generalized principal solution of the linear systems with some special matrices over the max-plus semiring.

A. Principal solution of the max linear system $T \bigodot x = u$ with $u \neq 0$

Theorem 5.1. *Let $T \in M_{c \times d}(\mathbf{G})$ be a regular row-wise aritmetic matrix with constant term 'a' and difference 'k' and $u_i = r + (i-1)dk \ \forall\ 1 \leq i \leq c$, where $r \in \mathbb{R}$. Then the equation $T \bigodot x = u$ has a principal solution $x_j^* = r - jk - a$, $\forall\ 1 \leq j \leq d$.*

Proof: Given matrix T is a regular row-wise arithmetic. To solve the max linear system $T \bigodot x = u$ with $u_i = r + (i-1)dk$, for $1 \leq i \leq c$. In AN_{Tu}- matrix we note that $d_{i1} = r - a - k, \cdots, d_{id} = r - a - dk$, for $1 \leq i \leq c$. $\implies$ $d_{ij} = r - a - jk$, for $1 \leq j \leq d$. All the elements of the AN_{Tu}- matrix are boxed. We find $x_j^* = min(d_{ij})$, $1 \leq i \leq c$, $1 \leq j \leq d \implies x_j^* = r - jk - a$, $\forall\ 1 \leq j \leq d$. ■

Corollary 5.1. *Let $T \in M_{c \times d}(\mathbf{G})$ be a regular k^{th} row-wise aritmetic with difference 'k' $\in \mathbb{R}$ and $u_i = r + (i-1)dk$ $\forall\ 1 \leq i \leq c$, where $r \in \mathbb{R}$. Then the equation $T \bigodot x = u$ has the principal solution $x_j^* = r - jk$, $\forall\ 1 \leq j \leq d$.*

Theorem 5.2. *Let $T \in M_{c \times d}(\mathbf{G})$ be a regular column-wise aritmetic with constant term 'a' and difference 'k' with $u_i = r + (i-1)k \ \forall\ 1 \leq i \leq c$, where $r \in \mathbb{R}$. Then the equation $T \bigodot x = u$ has the principal solution is $x_j^* = r - k - (j-1)ck - a$, $\forall\ 1 \leq j \leq d$.*

Proof: Given system $T \bigodot x = u$ is with the regular column-wise arithmetic and $u_i = r + (i-1)k$, $\forall\ 1 \leq i \leq c$. In AN_{Tu}-matrix we can conclude $d_{i1} = r - a - k, \cdots, d_{id} = r - a - (d-1)ck - k$, for $1 \leq i \leq c$ and every element of the AN_{Tu}-matrix is boxed. $\implies x_j^* = r - k - (j-1)ck - a$,$\forall\ 1 \leq j \leq d$. ■

Corollary 5.2. *Let $T \in M_{c \times d}(\mathbf{G})$ be a regular k^{th} column-wise arithmetic matrix and $T \bigodot x = u$ be a linear system over the max-plus semiring $(\mathbf{G})$. If the regular vector u of the form $u_i = r + (i-1)k$, $\forall\ 0 \leq i \leq c$ then the principal solution of the system is $x_j^* = r - k - (j-1)ck$, $\forall\ 1 \leq j \leq d$, for some $k \in \mathbb{R}$.*

Theorem 5.3. *Max linear system $T \bigodot x = u$ with $T = J_{c \times d}(l)(\mathbf{G})$ has a solution iff u is a constant vector.*

Proof: Assume that the system has a solution. It is enough to show that $u_i = u_j \ \forall\ 1 \leq i, j \leq c$. Suppose $u_i \neq u_j$ for some $1 \leq i, j \leq c$ and $u_i = r < u_j = k$ then i^{th} row of the AN_{Tu} matrix contains boxed element of every column. Similarly j^{th} row of AN_{Tu} matrix does not contains ny boxed element $\implies$ System never has a solution. Which is the contradiction to our assumption. Conversely assume u is a constant vector. Let $u_i = f$ for $1 \leq i \leq c$. Note that $d_{ij} = f - l$ and there is a box around $d_{ij} \ \forall\ 1 \leq i \leq c$,$\forall\ 1 \leq j \leq d \implies$ The system has a solution. ■

Corollary 5.3. *Max linear system $T \bigodot x = u$ with $T = J_{c \times d}(l)$ and u is a constant vector with the value f then the principal solution is $x_j^* = f - l$ for $1 \leq j \leq d$.*

Theorem 5.4. *Let $T \in M_{c\times d}(\mathbf{G})$ be a regular $\phi-$ diagonal matrix and $T \bigodot x = u$ be a linear system over the max-plus semiring $\mathbf{G}$ with the normal vector u. If $u_i = m$ is a constant vector then the system need not be have a solution.*

Proof: Assume that $u_i = m$ for $1 \le i \le c$. Note that $d_{kk} = m - (\phi)$ for $1 \le k \le c$ and all other elements are equal to m. We split this proof in two cases.

1) if $m < m - (\phi)$ then there is no box around d_{kk} , $\forall$ $1 \le k \le c$ and all other entries are boxed element $\implies$ system has a solution.
2) if $m > m - (\phi)$ and $c > d$ then d_{kk} are the boxed element, $\forall$ $1 \le k \le c$ and all other entries are unboxed. Last row of AN_{Tu} does not contain any boxed element $\implies$ system has no solution.

■

Theorem 5.5. *Let $T \in M_{c\times d}(\mathbf{G})$ be a regular $\phi-$ diagonal matrix and $T \bigodot x = u$ be a linear system over the max-plus semiring $\mathbf{G}$ with the normal vector u. If u is a constant vector as $u_i = m$, for $1 \le i \le c$ with $m - (\phi) > m$ then the principal solution is $x_j^* = m$, for $1 \le j < d$.*

Proof: Let us assume $u_i = m$, $1 \le i \le c$. We seperate this proof in three cases. $d_{ii} = m - (\phi)$, $1 \le i \le c$ and all other entries are m. If $m - (\phi) > m$ then d_{ii}'s are boxed elements for all $1 \le i \le c$ and d_{ij}'s are unboxed $\forall$ $i \ne j$.

1) If $c < d$ then all entries of last column of AN_{Tu} is boxed $\implies$ principal solution is $x_i^* = m$ for all $1 \le i \le d$.
2) If $c > d$ then every element in the last row of AN_{Tu} is unboxed $\implies$ principal solution for this case will be obtained by $x_i^* = m$, for all $1 \le i \le d$.
3) If $c = d$ then diagonal entries of AN_{Tu} matrix are unboxed all other elements of AN_{Tu} matrix is boxed $\implies$ $x_i^* = m$, for all $1 \le i \le d$.

■

Theorem 5.6. *Let $T \in M_{c\times d}(\mathbf{G})$ be a regular $\phi-$ diagonal matrix and $T \bigodot x = u$ be a linear system over the max-plus semiring $\mathbf{G}$ with the normal vector u. If u is a constant vector say $u_i = m$, for all $1 \le i \le c$ with $m - (\phi) < m$ and $c \le d$ then the principal solution is $x_j^* = m - \phi$, for all $1 \le j < d$.*

Proof: Given $m - \phi < m$. We go with 2 cases

1) If $c < d$ then $d_{ii} = m - \phi$ and $d_{ij} = m$ for $i \ne j$ $\implies$ each d_{ii}'s are boxed and d_{ij}'s are unboxed for all $i \ne j$ $\implies$ $x_j^* = m - \phi$, for all $1 \le j < d$.
2) If $c = d$ then $d_{ii} = m - \phi$ and $d_{ij} = m$, for $i \ne j$ $\implies$ $x_j^* = m - \phi$ for $1 \le j < d$.

■

Corollary 5.4. *Let $T \in M_{c\times c}(\mathbf{G})$ be a regular $\phi-$ diagonal matrix and $T \bigodot x = u$ be a linear system over the max-plus semiring $\mathbf{G}$ with the normal vector u. If u is a constant vector then the system has a solution.*

Theorem 5.7. *Let $T \bigodot x = u$ be a linear system with the regular arithmetic circulant matrix T over the max-plus semiring $(\mathbf{I})$. If $u_i = r+(i-1)k$, for $k \in \mathbb{R}$ and $r \in \mathbb{R}$. Then the system has a principal solution $x_j^* = r - k$ for $j = 1$ and $x_j^* = r - (c - l)k$, $\forall$ $0 \le l \le (c - 2)$, $\forall$ $2 \le j \le d$.*

Proof: In AN_{Tu} matrix 1^{st} column contains n times $r - 1$, 2^{nd} column contains one time $r - c$ and c-1 times r, similarly last column contains c-1 times $r - (c - (c - 2))$ and one time $r + (c - 2)$. $\implies$ $x_i^* = r - k$, for $i = 1$, $x_i^* = r - (c - l)k$, for $0 \le l \le (c - 2)$, $i \ne 1$.

■

B. Principal solution of the max linear system $T \bigodot x = 0$

Theorem 5.8. *If $T \bigodot x = 0$ be a linear system with the regular matrix $T \in M_{c\times d}(S)$ over the max-plus semiring S then $AN_{Tu} = -T$.*

Proof: $AN_j = u - T_j$, since AN_j denotes j^{th} column of AN_{Tu} matrix. Given u is zero vector $\implies$ $AN_j = -T_j$ $\implies$ $AN_{Tu} = -T$. ■

Theorem 5.9. *If $T \bigodot x = 0$ be a max linear system with $c \times d$ regular row-wise arithmetic matrix then the system has no solution.*

Proof: We know that $AN_{Tu} = -T$. Note that d_{ij}'s are boxed for $i = c$, $1 \le j \le d$ and d_{ij}'s are unboxed for $1 \le i \le (c - 1)$, $1 \le j \le d$ $\implies$ by Theorem 3.1 system has no solution. ■

Theorem 5.10. *If $T \bigodot x = 0$ be a max linear system with $c \times d$ regular column-wise arithmetic matrix then the system has no solution.*

Proof: In AN_{Tu} matrix, note that d_{ij}'s are boxed for $i = c$, $1 \le j \le d$ and d_{ij}'s are unboxed for $1 \le i \le (c-1)$, $1 \le j \le d$ $\implies$ by Theorem 3.1 system has no solution. ■

Corollary 5.5. *If $T \in M_{c\times c}(\mathbf{G})$ be a regular k^{th} row-wise arithmetic matrix and $T \bigodot x = 0$ is a max linear system then the system has no solution.*

Corollary 5.6. *If $T \in M_{c\times d}(\mathbf{F})$ be a regular k^{th} column-wise arithmetic matrix and $T \bigodot x = 0$ is the max linear system then the system has no solution.*

Theorem 5.11. *If $T \in M_{c\times d}(\mathbf{G})$ be a regular $J_{c\times d}(l)$ matrix over the max-plus semiring $\mathbf{G}$ and $T \bigodot x = 0$ be a linear system then the system has the principal solution is $x_j^* = -l$ for $1 \le j \le d$.*

Proof: $d_{ij} = -l$ for $1 \le i \le c$, $1 \le j \le d$ and $r_{ij} = 1$ for $1 \le i \le c$, $1 \le j \le d$ $\implies$ system has a solution. $\implies$ $x_j^* = -l$, for $1 \le j \le d$. ■

Theorem 5.12. *If $T \in M_{c\times d}(\mathbf{G})$ be a regular $\phi-$ diagonal matrix and $T \bigodot x = 0$ be a linear system over the max-plus semiring $\mathbf{G}$ then the system need not be have a solution.*

Proof: $d_{ii} = -\phi$ for $1 \le i \le c$ and $d_{ij} = 0$ otherwise. We split this proof by two parts,

1) If $-\phi < 0$ then the last row of AN_{Tu} does not contain any boxed $\implies$ system has no solution.
2) If $-\phi > 0$ then d_{ii}'s are unboxed, for $1 \le i \le c$ and all other entries are boxed $\implies$ every row of AN_{Tu} has atleast one boxed element $\implies$ system has a solution.

■

Theorem 5.13. *If $T \in M_{c\times d}(\mathbf{G})$ be a regular $\phi-$ diagonal matrix with $0 < -\phi$ and $T \bigodot x = 0$ be a linear system over the max-plus semiring $\mathbf{G}$ then the principal solution is $x_j^* = 0$, for $1 \le j \le d$.*

Proof: $d_{ii} = -\phi$ for $1 \le i \le c$ and $d_{ij} = 0$ otherwise. We split this proof by two parts,

1) If $c < d$ then every row of AN_{Tu} contains atleast one boxed element $\implies$ $x_j^* = 0$ for $1 \le j \le d$.

2) *If $c > d$ then every row of AN_{Tu} contains an boxed element atleast once $\implies x_j^* = 0$ for $1 \le j \le d$.*

■

Corollary 5.7. *If $T \in M_{c\times c}(\mathbf{G})$ be a regular $\phi-$ diagonal matrix and $T \bigodot x = 0$ be a linear system over the max-plus semiring $\mathbf{G}$ then the system has the principal solution is either $x_j^* = -(\phi)$ or $x_j^* = 0$, for $1 \le j \le c$.*

Proof: $d_{ii} = -\phi$ for $1 \le i \le c$ and $d_{ij} = 0$ otherwise. We split this proof by two parts,

1) *If $-\phi < 0$ then AN_{Tu} matrix contains its boxed elements in main diagonal and all other elements are unboxed $\implies$ system has a solution and $x_j^* = -\phi$.*
2) *If $-\phi > 0$ then d_{ii}'s are unboxed, for $1 \le i \le c$ and all other entries are boxed elements $\implies x_j^* = 0$, $1 \le j \le c$.*

■

Theorem 5.14. *If $T \bigodot x = 0$ be a max linear system with the regular column- wise arithmetic circulant matrix $T \in M_{c\times c}(\mathbf{F})$ then the system has a single solution and $x_j^* = -c$ for $1 \le j \le c$.*

Proof: Every row of AN_{Tu} matrix contains exactly one boxed element $\implies$ system has a single solution $\implies$ $x_j^* = -c$, $1 \le j \le c$. ■

VI. Method for finding free and leading variables by advanced normalization method

In this section we will introduce a method to find free and leading variables for given any max linear system $T \bigodot x = u$ that has a solution using the advanced normalization method. Once we found the leading variables all other variables are free variables. We will find free and leading variables by using the following steps.

1) Step 1 : For any given max linear system $T \bigodot x = u$ find the AN_{Tu} matrix by the advanced normalization method.
2) Step 2 : Find the rows of AN_{Tu} matrix which has exactly one boxed element then note the columns which the boxed elements are located. If the boxed elements are located in varies columns and varies rows then note all the columns say $j_1, j_2, j_3, ..., j_k$ and leading variables are $x_{j_1}, x_{j_2}, x_{j_3}, ..., x_{j_k}$ and delete the corresponding rows.
3) Step 3 : In the case of two rows say i and k has exactly one boxed element and boxed element of i^{th} row placed in j^{th} column and boxed element of k^{th} row placed in l^{th} column. Suppose the highest frequency column is l (the column which has more boxed element) and choose that x_l is a leading variable.
4) Step 4 : We have to construct AN_{Tu1} matrix with remaing rows. Suppose AN_{Tu1} matrix has any row which contains exactly one boxed element then repeat from Step 2. Otherwise go to Step 5.
5) Step 5 : In AN_{Tu1} matrix Step 2 and Step 3 will be repeated for all other cases which the remaining rows contains two or more boxed elements. After deleting all the rows of AN_{Tu} matrix we can find all the leading variables of the given max linear system. We can also find the number of free variables and leading variables by the formula "n = Number of free variable + Number of leading varible"

Example 6.1. *Consider the max linear system*

$$\begin{bmatrix} -1 & -2 & 15 & 0 & 3 \\ 6 & 5 & 11 & 6 & 2 \\ -6 & 4 & 9 & 3 & 5 \\ 5 & 11 & -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ 7 \\ 12 \end{bmatrix}$$

$$AN_{Tu} = \begin{bmatrix} 9 & \boxed{-2} & \boxed{-7} & \boxed{2} & 5 \\ \boxed{7} & 8 & 2 & 7 & 11 \\ 13 & 3 & -2 & 4 & \boxed{2} \\ \boxed{7} & 1 & 14 & 8 & 12 \end{bmatrix}$$

$$AN_{Tu}(1) = \begin{bmatrix} 9 & \boxed{-2} & \boxed{-7} & \boxed{2} & 5 \\ 13 & 3 & -2 & 4 & \boxed{2} \end{bmatrix}$$

$$AN_{Tu}(2) = \begin{bmatrix} 9 & \boxed{-2} & \boxed{-7} & \boxed{2} & 5 \end{bmatrix}$$

We will find the leading and free variables by using the advanced normalization method

- *In this example third, fourth and fifth rows contains exactly one boxed element. The boxed elements are placed in first and last columns. First column contains highest frequency boxed element (7). Remove every row which contains boxed element (7) $\implies x_1$ is a leading variable.*
- *In $AN_{Tu}(1)$ matrix last row contain exactly one boxed element which is placed in the last column. Delete last row and x_5 is a leading variable.*
- *Now in $AN_{Tu}(2)$ matrix first row contains three boxed elements in same frequency and x_2 is a leading variable.*
- *We have 3 leading variables (x_1, x_2, x_5) remaing variables are the free variable (x_3, x_4).*
- *DOF=2.*

VII. Conclusion and Future work

In this paper we introduced advanced normalization method for solving the max linear system. We developed three algorithms for solving a max linear system using the advanced normalization method. The method for finding the leading and free variables of any solvable max linear system using advanced normalization method is introduced. The sole purpose of Algorithm 1 is to determine whether the specified max linear system has a solution or not. If the system has a solution, Algorithm 2 is used to determine the possiblity of number of solutions. It can be a single solution or maybe infinitely many solutions. We can conclude the principal solution of the max linear system using Algorithm 3. We used the principal solution to obtain all other solutions in the case system attains many solutions. We solved the max linear system $T \bigodot x = u$ in both cases $u \ne 0$ and $T \bigodot x = 0$ with the assistance of the advanced normalization approach. Several crucial results are stated on the generalized principal solutions of max linear systems with particular matrices such as row-wise arithmetic matrix, column-wise arithmetic matrix, ϕ- diagonal matrix, $J_{n\times n}(l)$ matrix, arithmetic circulant. Our question was that the linear system $Tx = 0$ always have a solution or not. We answered that the linear system $Tx = 0$ does not always have a solution. In future we apply obtained results based on the solutions of tropical linear system in cryptography. Also these results can be applied to attack the key exchange protocols based on max linear systems over row-wise arithmetic matrices, column-wise arithmetic matrices, ϕ- diagonal matrices, $J_{c\times d}(l)$ matrices, arithmetic circulant matrices.

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